Tutorial # 1

N 4.1.15

Want to prove: $|z_1+z_2| \le |z_1|+|z_2|$ – triangle inequality

consider

$$|z_1+z_2|^2 = (z_1+z_2)\cdot(z_1+z_2)^* =$$

$$= (z_1+z_2)(z_1^*+z_2^*) = |z_1|^2 + \underbrace{z_1 z_2^* + \overbrace{z_1^* z_2}^{=(z_1 z_2^*)^*}}_{=2\cdot Re(z_1 \cdot z_2^*) \le 2|z_1\cdot z_2^*|} + |z_2|^2 \le$$

as sum of 2 complex conjugated numbers

$$\le |z_1|^2 + 2|z_1|\overbrace{|z_2^*|}^{=|z_2|} + |z_2|^2 = (|z_1|+|z_2|)^2$$

Taking square roots of both sides, we obtain:

$$|z_1+z_2| \le |z_1|+|z_2|$$

N 4.3.8

$$\left(\frac{1-i}{1+i}\right)^8 = \left(\frac{(1-i)^2}{(1+i)(1-i)}\right)^8 = \left(\frac{1-2i-1}{1+1}\right)^8 =$$

$$=(-i)^8 = i^8 = (i^2)^4 = (-1)^4 = 1$$

N 9.1.8

subtracted 3^{rd} row

$$\begin{vmatrix} x & 1 & 0 & 1 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 1 & 0 & 1 & x \end{vmatrix} \underset{\text{using properties of determinant}}{=} \begin{vmatrix} x & 0 & -x & 0 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 0 & -x & 0 & x \end{vmatrix} \underset{\text{do cofactor expansion}}{=}$$

subtracted 2^{nd} row

$$= \cancelto{1}{(-1)^{1+1}} \cdot x \cdot \begin{vmatrix} x & 1 & 0 \\ 1 & x & 1 \\ -x & 0 & x \end{vmatrix} - \cancelto{1}{(-1)^{1+3}} x \cdot \begin{vmatrix} 1 & x & 0 \\ 0 & 1 & 1 \\ 0 & -x & x \end{vmatrix} =$$

$$= x \cdot [x^3 - (x+x)] - x \cdot [x+x] =$$

$$= x^2(x^2-4) = 0 \iff \left[\begin{array}{l} x = 0 \\ x = \pm 2 \end{array}\right.$$

№ 9.2.11

$$\begin{cases} x+y+z=6 \\ x+\lambda y+\lambda z=2 \end{cases}$$ – underdetermined system $\Rightarrow$ there can't be unique solution!

Formally, write augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & \lambda & \lambda & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & \lambda-1 & \lambda-1 & -4 \end{array}\right) \Rightarrow$$

$$\Rightarrow \left[\begin{array}{ll} \lambda = 1 & \text{– no solutions } (0 \neq -4\,!) \\ \lambda \neq 1 & \text{– infinitely many solutions} \\ & \text{(system is underdetermined)} \end{array}\right.$$

This is obvious result since, clearly we can solve the system for x, $\tilde{y} := y+z$. It will have a unique solution when $\lambda \neq 1$

$$x = 6 + \frac{4}{\lambda-1}, \quad \tilde{y} = -\frac{4}{\lambda-1},$$ but

there're infinitely many y, z satisfying $y+z = -\frac{4}{\lambda-1}$

$\sim$ 9.2.8 / 9.4.11

$$\begin{cases} x_1 - 2x_2 + 3x_3 - x_4 = 0 \\ -x_1 + 2x_3 + x_4 = 0 \\ 2x_1 + x_2 - 2x_4 = 0 \end{cases} \iff \begin{pmatrix} 1 & -2 & 3 & -1 \\ -1 & 0 & 2 & 1 \\ 2 & 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

$$A := \begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & -2 & 5 & 0 \\ 0 & 5 & -6 & 0 \end{pmatrix} \sim$$

added 1^{st} row

subtracted $2 \times (1^{st}$ row$)$

$$\sim \begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & -2 & 5 & 0 \\ 0 & 0 & -6+\frac{25}{2} & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 - 2x_2 + 3x_3 - x_4 = 0 \\ -2x_2 + 5x_3 = 0 \\ \frac{13}{2} \cdot x_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_3 = 0 \\ x_2 = 0 \\ x_1 = x_4 \end{cases}$$

x_4 – arbitrary

– infinitely many solutions !

it's always the case for underdetermined homogeneous system

zero-column (system is homogeneous)

$$\text{rank } A = \text{rank} \underbrace{(A \mid H)}_{\text{augmented matrix}} = 3 < \dim(x_1, x_2, x_3, x_4) = 4$$

N 9.6. 16

$v_1 = (1, -1, 0)$, $v_2 = (1, 1, 0)$, $v_3 = (0, 1, 1)$

We will ~~first~~ form orthogonal basis.

Let $u_1 := v_1$

$u_2 := v_2 + a \cdot u_1$, $a \in \mathbb{R}$

We want $\langle u_2, u_1 \rangle = 0 \Rightarrow \underbrace{\langle v_2, v_1 \rangle}_{= 1 - 1 + 0 = 0} + a \cdot \|v_1\|^2 = 0$

$\Rightarrow a = 0$ (v_1, v_2 are already orthogonal)

$\Rightarrow u_2 = v_2$

Let $u_3 := v_3 + a_1 \cdot u_1 + a_2 \cdot u_2$

$$\begin{cases} \langle u_3, u_1 \rangle = 0 \\ \langle u_3, u_2 \rangle = 0 \end{cases} \Rightarrow \begin{cases} \langle v_3, v_1 \rangle + a_1 \cdot \|v_1\|^2 = 0 \\ \langle v_3, v_2 \rangle + a_2 \cdot \|v_2\|^2 = 0 \end{cases}$$

$$\Rightarrow a_1 = -\frac{\langle v_3, v_1 \rangle}{\|v_1\|^2} = \frac{1}{2}$$

$$a_2 = -\frac{\langle v_3, v_2 \rangle}{\|v_2\|^2} = -\frac{1}{2}$$

Hence, $u_3 = \left(0 + \frac{1}{2} - \frac{1}{2}, 1 - \frac{1}{2} - \frac{1}{2}, 1 + 0 + 0\right) =$

$= (0, 0, 1)$

Note u_3 is already normalized so let's normalize u_1, u_2:

$u_1^n := \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2}}(1, -1, 0)$, $u_2^n := \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{2}}(1, 1, 0)$, $u_3^n = (0, 0, 1)$

N 9.7.6

We have: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

We want to show the equation

$$C_1 \cdot I + C_2 \cdot \sigma_x + C_3 \cdot \sigma_y + C_4 \cdot \sigma_z = 0$$

holds true only if $C_1 = C_2 = C_3 = C_4 = 0$.

Indeed, $\begin{pmatrix} C_1 + C_4 & C_2 - iC_3 \\ C_2 + iC_3 & C_1 - C_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$\begin{cases} C_1 + C_4 = 0 \\ C_1 - C_4 = 0 \\ C_2 - iC_3 = 0 \\ C_2 + iC_3 = 0 \end{cases}$$

$\Rightarrow C_1 = C_4 = 0$

$\Rightarrow C_2 = C_3 = 0$

$\Rightarrow I, \sigma_x, \sigma_y, \sigma_z$ are linearly independent

N 9.7.14

Kronecker delta

$\{\varphi_j\}_{j=1,\dots,n}$ — orthonormal, i.e. $\langle \varphi_i, \varphi_j \rangle = \delta_{ij} := \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$

Let $v := c_1 \cdot \varphi_1 + c_2 \cdot \varphi_2 + \dots + c_n \varphi_n = \sum_{k=1}^{n} c_k \varphi_k$

Then, multiplying this by φ_j, we get

$$\langle v, \varphi_j \rangle = \sum_{k=1}^{n} c_k \cdot \underbrace{\langle \varphi_k, \varphi_j \rangle}_{=\delta_{kj}} = c_j$$

Consider now smaller span: $\sum_{j=1}^{r} c_j \cdot \varphi_j$, $r \leq n$.

Introduce $u := v - \sum_{j=1}^{r} c_j \varphi_j$.

Squaring both sides, we have:

$$\|u\|^2 = \|v\|^2 + \sum_{j=1}^{r} c_j^2 - 2 \sum_{j=1}^{r} \underbrace{\langle v, \varphi_j \rangle}_{=c_j} \cdot c_j =$$

$$= \|v\|^2 - \sum_{j=1}^{r} c_j^2 \geq 0$$

$$\Rightarrow \quad \|v\|^2 \geq \sum_{j=1}^{r} c_j^2 \quad \text{— Bessel's inequality}$$