Tutorial # 10

$\sim 16.1.4$

Let $t \geq 0$, $x \in \mathbb{R}$.

$$\frac{\partial C}{\partial t} = \overset{\text{const}}{\overset{\shortparallel}{D}} \cdot \underbrace{\Delta C}_{= \nabla^2 C = \frac{\partial^2 C}{\partial x^2}}$$ - diffusion / heat eq.

$$C(x,t) = t^{-1/2} \cdot \exp\left(-x^2/4Dt\right)$$ - solution to verify

$$\frac{\partial C}{\partial t} = \left(-\frac{1}{2}\cdot\frac{1}{t} + \frac{x^2}{4Dt^2}\right)\cdot t^{-1/2}\cdot \exp\left(-x^2/4Dt\right)$$

$$\frac{\partial^2 C}{\partial x^2} = \frac{\partial}{\partial x}\left[-\frac{2x}{4Dt}\cdot t^{-1/2}\cdot \exp\left(-x^2/4Dt\right)\right] =$$

$$= \left(-\frac{1}{2Dt} + \frac{4x^2}{16D^2t^2}\right)\cdot t^{-1/2}\cdot \exp\left(-x^2/4Dt\right)$$

$$\Rightarrow \quad \frac{\partial C}{\partial t} = D\cdot\frac{\partial^2 C}{\partial x^2}$$

$C(x,0)$ - ?

For $x = 0$: $C(0,t) = t^{-1/2} \xrightarrow[t\to 0+]{} +\infty$

For $x \neq 0$:

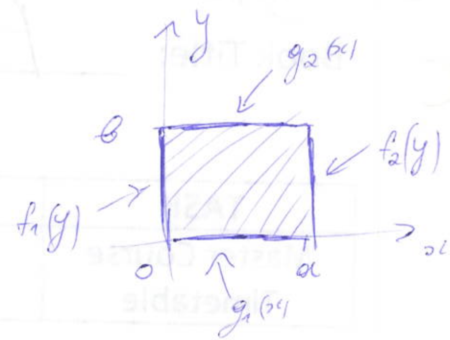
$$\lim_{t\to 0+} C(x,t) = \lim_{t\to 0+}\frac{\exp\left(-x^2/4Dt\right)}{t^{1/2}} = \lim_{t\to 0+}\frac{t^{-1/2}}{\exp\left(x^2/4Dt\right)} =$$ (use L'Hopital's rule)

$$= \lim_{t\to 0+}\frac{-\frac{1}{2}t^{-3/2}}{-\frac{x^2}{4D}t^{-2}\cdot\exp\left(x^2/4Dt\right)} =$$

$$= \lim_{t\to 0+}\frac{2D}{x^2}\cdot t^{1/2}\cdot\exp\left(-x^2/4Dt\right) = 0$$

№ 16.2.6

$$(*)\begin{cases} \Delta u := \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & x \in (0, a) \\ & y \in (0, b) \\ u(0,y) = f_1(y), \quad u(a,y) = f_2(y), \\ u(x,0) = g_1(x), \quad u(x,b) = g_2(x), \end{cases}$$



Let $u(x,y) = v(x,y) + w(x,y)$ where v, w are such that

$$(1)\begin{cases} \Delta v = 0 \\ v(0,y) = f_1(y), \ v(x,0) = 0 \\ v(a,y) = f_2(y), \ v(x,b) = 0 \end{cases} \qquad (2)\begin{cases} \Delta w = 0 \\ w(x,0) = g_1(x), \ w(0,y) = 0 \\ w(x,b) = g_2(x), \ w(a,y) = 0 \end{cases}$$

Then, by linearity $(*) = (1) + (2)$. That is if we can solve (1) and (2) the sum of the solutions will be a solution to $(*)$.

Each of (1), (2) can be solved by separation of variables. For instance, consider (1).

Let $v(x,y) = X(x) \cdot Y(y)$. Then:

$$X''Y + XY'' = 0 \overset{:(XY)}{\Longrightarrow} \underbrace{\frac{X''}{X}}_{\text{function of } x \text{ alone}} = -\underbrace{\frac{Y''}{Y}}_{\text{function of } y \text{ alone}} = \underbrace{\lambda}_{\text{constant}}$$

$$\begin{cases} Y'' + \lambda Y = 0 \\ Y(0) = 0, \quad Y(b) = 0 \end{cases}$$

$Y(y) = C_1 \cdot \cos(\sqrt{\lambda} y) + C_2 \sin(\sqrt{\lambda} y)$

$Y(0) = 0 \Rightarrow C_1 = 0$; $Y(b) = 0 \Rightarrow \sin(\sqrt{\lambda} b) = 0 \Rightarrow \sqrt{\lambda} b = \pi n$

$\lambda_n = \dfrac{\pi^2 n^2}{b^2} > 0$

$$X'' - \lambda X = 0 \quad \Rightarrow \quad X(x) = C_3 \cdot e^{-\sqrt{\lambda}x} + C_4 \cdot e^{\sqrt{\lambda}x}$$

$$v(x,y) = \sum_{n=1}^{\infty} X_n(x) \cdot Y_n(y) = \sum_{n=1}^{\infty} \left[A_n \cdot e^{-\frac{\pi n x}{b}} + B_n \cdot e^{\frac{\pi n x}{b}} \right] \cdot \sin\left(\frac{\pi n y}{b}\right)$$

$$v(0,y) = f_1(y) = \sum_{n=1}^{\infty} \left(A_n + B_n\right) \cdot \sin\left(\frac{\pi n y}{b}\right)$$

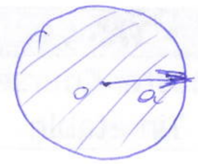
$$v(a,y) = f_2(y) = \sum_{n=1}^{\infty} \left(A_n e^{-\frac{\pi n a}{b}} + B_n e^{\frac{\pi n a}{b}}\right) \cdot \sin\left(\frac{\pi n y}{b}\right)$$

$$\Rightarrow \begin{cases} A_n + B_n = \dfrac{2}{b} \displaystyle\int_0^b f_1(y) \cdot \sin\left(\frac{\pi n y}{b}\right) dy \\ A_n e^{-\frac{\pi n a}{b}} + B_n e^{\frac{\pi n a}{b}} = \dfrac{2}{b} \cdot \displaystyle\int_0^b f_2(y) \cdot \sin\left(\frac{\pi n y}{b}\right) dy \end{cases}$$

$$\Rightarrow \begin{cases} A_n = \dots \\ B_n = \dots \end{cases}, \; n = 1, 2, \dots \quad \Rightarrow \quad v(x,y) = \sum_{n=1}^{\infty} \left(A_n e^{-\frac{\pi n x}{b}} + B_n e^{\frac{\pi n x}{b}}\right) \cdot \sin\left(\frac{\pi n y}{b}\right) = \dots$$

(~ 16-2.11)

$$\begin{cases} \Delta u(r,\theta) = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\cdot\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 \le r < a \\ u(a,\theta) = f(\theta) \end{cases}$$



$$u(r,\theta) = R(r)\cdot Y(\theta)$$

$$R''Y + \frac{1}{r}R'Y + \frac{1}{r^2}RY'' = 0$$

$$\Rightarrow \frac{R''}{R} + \frac{1}{r}\cdot\frac{R'}{R} + \frac{1}{r^2}\frac{Y''}{Y} = 0$$

$$\underbrace{r^2\frac{R''}{R} + r\frac{R'}{R}}_{\text{function of } r} = \underbrace{-\frac{Y''}{Y}}_{\text{function of } \theta} = \lambda$$

$$\begin{cases} Y'' + \lambda Y = 0 \\ Y(\theta) = Y(2\pi+\theta) \end{cases} \Rightarrow$$

$\lambda \geq 0$

$$Y(\theta) = C_1\cos(\sqrt{\lambda}\,\theta) + C_2\sin(\sqrt{\lambda}\,\theta)$$

$$Y(2\pi+\theta) = C_1\cos(\sqrt{\lambda}\,\theta + \sqrt{\lambda}\,2\pi) + C_2\sin(\sqrt{\lambda}\,\theta + \sqrt{\lambda}\,2\pi)$$

$$\sqrt{\lambda}\cdot 2\pi = 2\pi n \Rightarrow \lambda_n = n^2$$

$$r^2R'' + rR' - \lambda R = 0$$

Try $R(r) = r^\alpha$. Then: $(\alpha(\alpha-1) + \alpha - \lambda)\cdot r^\alpha = 0$

$$\Rightarrow \alpha^2 = \lambda \Rightarrow \alpha = \pm\sqrt{\lambda}$$

Since $R(0) < \infty$ (for $u(r,\theta)$ to be defined in the center of the disk) we choose $\alpha = \sqrt{\lambda}$

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty}\left(C_n \cdot r^n \cdot \cos(n\theta) + D_n\, r^n \cdot \sin(n\theta)\right)$$

$$u(a,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n \cdot \left[C_n \cdot \cos(n\theta) + D_n \cdot \sin(n\theta)\right] = f(\theta)$$

$$C_n = \frac{1}{a^n} \cdot \overbrace{\frac{1}{\pi} \cdot \int_0^{2\pi} f(\theta) \cdot \cos(n\theta)\, d\theta}^{=:\, a_n}, \quad n = 0, 1, 2, \ldots$$

$$D_n = \frac{1}{a^n} \cdot \underbrace{\frac{1}{\pi} \cdot \int_0^{2\pi} f(\theta) \cdot \sin(n\theta)\, d\theta}_{=:\, b_n}, \quad n = 1, 2, 3, \ldots$$

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cdot \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right]$$

№ 16.2.12

$$\underbrace{\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos\left[n(\theta - x)\right]} = \underbrace{\frac{a^2 - ar\cos(\theta - x)}{a^2 - 2ar \cdot \cos(\theta - x) + r^2} - 1}_{\text{want to show}}$$

$$= \operatorname{Re}\left\{\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cdot e^{in(\theta - x)}\right\} = \operatorname{Re}\sum_{n=1}^{\infty}\left(\frac{r}{a} \cdot e^{i(\theta - x)}\right)^n =$$

$$= \frac{1 - \frac{r}{a} e^{-i(\theta - x)}}{1 + (r/a)^2 - 2\frac{r}{a}\cos(\theta - x)}$$

for $r < a$

$$= \operatorname{Re}\sum_{n=0}^{\infty}\left(\frac{r}{a} e^{i(\theta - x)}\right)^n - 1 = \operatorname{Re}\left(\frac{1}{1 - \frac{r}{a} e^{i(\theta - x)}}\right) - 1 =$$

$$= \frac{1 - \frac{r}{a}\cos(\theta - x)}{1 + \left(\frac{r}{a}\right)^2 - 2\left(\frac{r}{a}\right)\cos(\theta - x)} - 1 = \frac{a^2 - ar\cos(\theta - x)}{a^2 - 2ar\cos(\theta - x) + r^2} - 1$$

№ 16.2.13

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right] =$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)\, dx \quad ; \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\, dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx + \frac{1}{\pi} \int_0^{2\pi} \underbrace{\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cdot \underbrace{\left[\cos(nx) \cdot \cos(n\theta) + \sin(nx) \sin(n\theta)\right]}_{= \cos(n(\theta - x))}}_{= \frac{a^2 - ar\cos(\theta - x)}{a^2 - 2ar\cos(\theta - x) + r^2} - 1} f(x)\, dx =$$

(interchanging $\sum$ & $\int$)

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[2 \cdot \frac{a^2 - ar\cos(\theta - x)}{a^2 - 2ar\cos(\theta - x) + r^2} \underbrace{-2 + 1}_{= -1} \right] f(x)\, dx =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{2a^2 - 2ar\cos(\theta - x) - a^2 + 2ar\cos(\theta - x) - r^2}{a^2 - 2ar\cos(\theta - x) + r^2} \cdot f(x)\, dx =$$

$$= \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(x)}{a^2 - 2ar\cos(\theta - x) + r^2}\, dx$$

№ 16.2.14

Set $r = 0$:

$$\underbrace{u(0,\theta)}_{\text{value on the center}} = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx = \underbrace{\frac{1}{2\pi} \int_0^{2\pi} u(a,\theta)\, d\theta}_{\text{average over the Boundary}}$$