Tutorial #2

№ 10.1.11

Let A be orthogonal matrix, i.e. $A^TA = AA^T = I$.

Want to show $\det A = \pm 1$.

$$\det(A^TA) = \underbrace{\det(A^T)}_{=\det(A)} \cdot \det(A) = \det(I) = 1$$

by properties of determinant

on the other hand

$$\Rightarrow \left[\det(A)\right]^2 = 1 \Rightarrow \det A = \pm 1$$

№ 10.2.12

Let A be skew-Hermitian matrix, i.e. $\underbrace{A^+}_{=(A^*)^T} = -A$.

Want to show it has purely imaginary e-values.

Consider

$$\underbrace{\langle \underbrace{Ax}_{=\lambda x}, x \rangle}_{=\lambda^* \cdot \|x\|^2} = \langle x, \underbrace{A^+x}_{-Ax = -\lambda x} \rangle = -\lambda^* \|x\|^2$$

$$\Rightarrow \lambda^* + \lambda = 0 = 2\,Re(\lambda) \Rightarrow Re(\lambda) = 0$$

N 10.4.3

$$f(x,y) = \underbrace{\begin{pmatrix} x & y \end{pmatrix}}_{=: \vec{x}^T} \underbrace{\begin{pmatrix} a & b \\ b & c \end{pmatrix}}_{=: A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{x}}$$

$$= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} ax + by \\ bx + cy \end{pmatrix} = ax^2 + 2bxy + cy^2$$

old coordinates — new coordinates

Let $\vec{x} = R\,\vec{x}'$ i.e. rotate our coordinate system.

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

rotation through angle θ

Sub this into $f(x,y) = \vec{x}^T A \vec{x}$ to see how this quadratic form transforms under such coordinate transformation

$$g(x',y') = \underbrace{(R\vec{x}')^T}_{\vec{x}^T} A \underbrace{(R\vec{x}')}_{\vec{x}} = \vec{x}'^T R^T A R\, \vec{x}'$$

To eliminate the cross term in $g(x',y')$, we require $R^T A R$ be diagonal:

$$R^T A R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a\cos\theta + b\sin\theta & -a\sin\theta + b\cos\theta \\ b\cos\theta + c\sin\theta & -b\sin\theta + c\cos\theta \end{pmatrix} =$$

$$= \begin{pmatrix} a\cdot\cos^2\theta + 2b\sin\theta\cos\theta + c\cdot\sin^2\theta & b(\cos^2\theta - \sin^2\theta) + (c-a)\sin\theta\cos\theta \\ -b(\sin\theta - \cos^2\theta) + (c-a)\sin\theta\cos\theta & a\sin^2\theta - 2b\sin\theta\cos\theta + c\cdot\cos^2\theta \end{pmatrix}$$

This will be diagonal when:

$$b\underbrace{(\cos\theta - \sin\theta)}_{\cos 2\theta} = (a-c)\underbrace{\sin\theta\cos\theta}_{\frac{1}{2}\sin 2\theta}$$

$$\Rightarrow \tan 2\theta = \frac{2b}{a-c}$$

Note: it was possible to make $R^T A R$ diagonal by imposing one condition on θ because it's symmetric $\left((R^T A R)^T = R^T A^T R = R^T A R \right)$ that is always the case for a quadratic form matrix.

N 10.2.8

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be distinct e-values with $x_1, x_2, \dots, x_n$ e-vectors. Lets show that $x_1, x_2, \dots, x_n$ are linearly independent.

Consider span $y := c_1 x_1 + c_2 x_2 + \dots + c_n x_n$ and prove that $y = 0 \Rightarrow c_1 = c_2 = \dots = c_n = 0$.

First act on y by $(A - \lambda_2 I)$:

$$(A - \lambda_2 I) y = c_1 (\underbrace{A x_1}_{=\lambda_1 x_1} - \lambda_2 x_1) + \cancel{c_2 (\underbrace{A x_2}_{=\lambda_2 x_2} - \lambda_2 x_2)} + c_3 (\underbrace{A x_3}_{=\lambda_3 x_3} - \lambda_2 x_3) + \dots + c_n (\underbrace{A x_n}_{=\lambda_n x_n} - \lambda_2 x_n) =$$

$$= c_1 (\lambda_1 - \lambda_2) x_1 + \underbrace{c_3 (\lambda_3 - \lambda_2) x_3 + \dots + c_n (\lambda_n - \lambda_2) x_n}_{(n-2)\text{ terms}}$$

Now act on both sides by $(A - \lambda_3 I)$:

$$(A - \lambda_3 I)(A - \lambda_2 I) y = c_1 (\lambda_1 - \lambda_2)(\underbrace{A x_1}_{=\lambda_1 x_1} - \lambda_3 x_1) + \underbrace{\cancel{c_3 (\lambda_3 - \lambda_2)(\underbrace{A x_3}_{=\lambda_3 x_3} - \lambda_3 x_3)}}_{\to 0} + c_4 (\lambda_4 - \lambda_2)(\underbrace{A x_4}_{=\lambda_4 x_4} - \lambda_3 x_4) + \dots +$$

$$+ c_n (\lambda_n - \lambda_2)(\underbrace{A x_n}_{=\lambda_n x_n} - \lambda_3 x_n) = c_1 (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) x_1 + \underbrace{c_4 (\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3) x_4 + \dots + c_n (\lambda_n - \lambda_2)(\lambda_n - \lambda_3) x_n}_{(n-3)\text{ terms}}$$

Repeating this process $(n-1)$ times, we get rid of all terms in RHS ("right hand side") except the very first one $\sim x_1$:

$$(A-\lambda_n I)(A-\lambda_{n-1} I)\dots(A-\lambda_3 I)(A-\lambda_2 I)\,y =$$

$$= C_1 \cdot \underbrace{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\dots(\lambda_1-\lambda_{n-1})(\lambda_1-\lambda_n)}_{\neq 0 \text{ - since all e-values are distinct}}\; \underbrace{x_1}_{\neq 0 \text{ since it's e-vector}}$$

Now putting $y=0$, we conclude $C_1 = 0$.

Note: we acted on y by product of matrices not containing $(A-\lambda_1 I)$.

In a similar manner, we can repeat the same procedure with chain of matrices not containing $(A-\lambda_2 I)$.

$$(A-\lambda_n I)(A-\lambda_{n-1} I)\dots(A-\lambda_3 I)(A-\lambda_1 I)\,y =$$

$$= C_2 \cdot \underbrace{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)\dots(\lambda_2-\lambda_{n-1})(\lambda_2-\lambda_n)}_{\neq 0} \cdot \underbrace{x_2}_{\neq 0}$$

Then:

$$y = 0 \implies C_2 = 0$$

If we keep on going with this n times, we eventually obtain $C_1 = C_2 = \dots = C_n = 0$ that is required.