Tutorial # 3

N 10.3.15 — N 10.3.17

$$\begin{cases} 8\ddot{x}_1 = -8x_1 + 2x_2 \\ 2\ddot{x}_2 = -3x_2 + 2x_1 + x_3 \\ 2\ddot{x}_3 = -2x_3 + x_2 \end{cases} \Rightarrow \begin{cases} \ddot{x}_1 = -x_1 + \frac{1}{4}x_2 \\ \ddot{x}_2 = x_1 - \frac{3}{2}x_2 + \frac{1}{2}x_3 \\ \ddot{x}_3 = \frac{1}{2}x_2 - x_3 \end{cases}$$

Denote $\vec{x} = (x_1, x_2, x_3)^T$. Let $\vec{x} = \vec{u} \cdot e^{i\omega t}$

t-independent constant vector

Then:

$$\underset{-\omega^2 \cdot \vec{u} \cdot e^{i\omega t}}{\overset{\parallel}{\ddot{\vec{x}}}} = \underbrace{\begin{pmatrix} -1 & \frac{1}{4} & 0 \\ 1 & -3/2 & \frac{1}{2} \\ 0 & 1/2 & -1 \end{pmatrix}}_{=:A} \vec{x}$$

$= \vec{u} \cdot e^{i\omega t}$

$$\Rightarrow \quad \underbrace{A\vec{u} = -\omega^2 \vec{u}}$$

e-value problem (can relabel $\lambda = -\omega^2$)

$$\begin{vmatrix} -1+\omega^2 & 1/4 & 0 \\ 1 & -3/2+\omega^2 & 1/2 \\ 0 & 1/2 & -1+\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow (\omega^2-1)\left[(\omega^2-1)(\omega^2-3/2) - 1/4\right] - \frac{1}{4}(\omega^2-1) = 0$$

$$(\omega^2-1)\left[(\omega^2-1)(\omega^2-3/2) - 1/2\right] = 0$$

Fundamental frequencies:

(1): $\omega_1^2 = 1$

(2)–(3): $\omega^4 - 5/2\,\omega^2 + 1 = 0$

$$\omega_{2,3}^2 = \frac{5/2 \mp \sqrt{25/4 - 4}}{2} = 5/4 \mp 3/4$$

$$\omega_2^2 = 1/2 \quad , \quad \omega_3^2 = 2$$

Normal modes:

(1): $\omega_1^2 = 1$:

$$\begin{pmatrix} 0 & 1/4 & 0 \\ 1 & 0-1 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} u_2 = 0 \\ u_1 = -\frac{1}{2} u_3 \end{matrix} \quad \text{, i.e.} \quad \vec{u} = \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \quad \text{– e-vector}$$

Denoting $c := u_3 = A \cdot e^{i\varphi}$ (complex constant), we have

$$\begin{cases} x_1(t) = \mathrm{Re}\left(-c/2 \cdot e^{it}\right) = -\frac{A}{2} \cdot \cos(t + \varphi) = \frac{A}{2} \cdot \cos(t + \varphi + \pi) \\ x_2(t) = 0 \\ x_3(t) = \mathrm{Re}\left(c \cdot e^{it}\right) = A \cdot \cos(t + \varphi) = A \cdot \cos(t + \varphi) \end{cases}$$

(2): $\omega_2^2 = 1/2$:

$$\begin{pmatrix} -1/2 & 1/4 & 0 \\ 1 & -1 & 1/2 \\ 0 & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} u_1 = \frac{1}{2} u_2 \\ u_3 = u_2 \end{matrix} \quad \text{i.e.} \quad \vec{u} = \begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix} \text{ - e-vector}$$

Let $u_2 =: C = A e^{i\varphi}$. Then:

$$\begin{cases} x_1(t) = \mathrm{Re}\left(\frac{1}{2} C \cdot e^{\frac{it}{\sqrt{2}}}\right) = \frac{A}{2} \cos\left(t/\sqrt{2} + \varphi\right) \\ x_2(t) = \mathrm{Re}\left(C \cdot e^{\frac{it}{\sqrt{2}}}\right) = A \cdot \cos\left(t/\sqrt{2} + \varphi\right) \\ x_3(t) = \text{"} = A \cdot \cos\left(t/\sqrt{2} + \varphi\right) \end{cases}$$

(3): $\omega_3^2 = 2$:

$$\begin{pmatrix} 1 & 1/4 & 0 \\ 1 & 1/2 & 1/2 \\ 0 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} u_1 = -\frac{1}{4} u_2 \\ u_3 = -\frac{1}{2} u_2 \end{matrix} \Rightarrow \vec{u} = \begin{pmatrix} -1/4 \\ 1 \\ -1/2 \end{pmatrix}$$

(e-vector)

Let $u_2 =: C = A \cdot e^{i\varphi}$. Then:

$$\begin{cases} x_1(t) = A/4 \cdot \cos(\sqrt{2} t + \varphi + \pi) \\ x_2(t) = A \cdot \cos(\sqrt{2} t + \varphi) \\ x_3(t) = A/2 \cdot \cos(\sqrt{2} t + \varphi + \pi) \end{cases}$$

N 10. 3. 18

Consider total energy of the system:

$$E = \overset{\text{kinetic}}{T} + \overset{\text{potential}}{V} = \underbrace{8\,\frac{\dot{x}_1^2}{2} + 2\cdot\frac{\dot{x}_2^2}{2} + 2\cdot\frac{\dot{x}_3^2}{2}}_{=T} +$$

$$+ \underbrace{3x_1^2 + (x_1 - x_2)^2 + \frac{1}{2}(x_3 - x_2)^2 + \frac{1}{2}x_3^2}_{=V\text{ - given}}$$

Take the lowest frequency normal mode, i.e.

$$\begin{cases} x_1 = A/2 \cdot \cos\left(t/\sqrt{2} + \varphi\right) \\ x_2 = A \cdot \cos\left(t/\sqrt{2} + \varphi\right) \\ x_3 = A \cdot \cos\left(t/\sqrt{2} + \varphi\right) \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = -\frac{A}{2\sqrt{2}} \cdot \sin\left(t/\sqrt{2} + \varphi\right) \\ \dot{x}_2 = \dot{x}_3 = -\frac{A}{\sqrt{2}} \cdot \sin\left(t/\sqrt{2} + \varphi\right) \end{cases}$$

and plug it into E:

$$E = \frac{3}{2}A^2 \cdot \sin^2\left(t/\sqrt{2} + \varphi\right) + A^2 \cos^2\left(t/\sqrt{2} + \varphi\right)\underbrace{\left[\frac{3}{4} + \frac{1}{4} + 0 + \frac{1}{2}\right]}_{=3/2} =$$

$$= \frac{3}{2}A^2 \cdot \underbrace{\left[\sin^2\left(t/\sqrt{2} + \varphi\right) + \cos^2\left(t/\sqrt{2} + \varphi\right)\right]}_{=1} = \frac{3}{2}A^2$$

const independent of t

$\Leftarrow$ E is conserved

N 10.4.18

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{pmatrix}$$ a matrix in standard $(E)_{\text{basis}}$

$w_1 = (1, 0, 0)^T$, $w_2 = (0\ 1\ 2)^T$, $w_3 = (0\ 0\ 1)^T$

vectors given in E-basis

$u_E = (3\ 0\ 2)^T$

Let $v_E = A u_E$. We want to find v_w, i.e. the components of the same vector in basis formed by w_1, w_2, w_3

$$v_E = W v_w \Rightarrow v_w = W^{-1} v_E$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}}_{\text{components of } w\text{-basis vectors in E-basis}} \Rightarrow W^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$ as transposed cofactor matrix since $\det W = 1$

$$v_w = W^{-1} v_E = W^{-1} A u_E = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{pmatrix}}_{= \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -2 \\ -3 & 0 & 5 \end{pmatrix}} \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \\ 1 \end{pmatrix}$$

N 10. 6. 15

$$I = \int_{-\infty}^{+\infty} e^{itx - \frac{1}{2}hx^2} dx = \int_{-\infty}^{+\infty} e^{itx} \cdot e^{-\frac{1}{2}hx^2} dx =$$

$$e^{itx} = \sum_{k=0}^{\infty} \frac{(itx)^k}{k!} = \sum_{k=0}^{\infty} \frac{(itx)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(itx)^{2k}}{(2k+1)!}$$

Taylor expansion

$$= \sum_{k=0}^{\infty} \int_{-\infty}^{+\infty} \frac{(itx)^{2k}}{(2k)!} \cdot e^{-\frac{1}{2}hx^2} dx =$$

Since $\int_{-\infty}^{+\infty} \underbrace{\underbrace{x^{2k+1}}_{odd} \cdot \underbrace{e^{-\frac{1}{2}hx^2}}_{even}}_{odd} dx = 0$ for $\forall k$

Compute separately: $\int_{-\infty}^{+\infty} x^{2k} \cdot e^{-\overbrace{\frac{h}{2}}^{=:\lambda} \cdot x^2} dx = (-1)^k \cdot \frac{d^k}{d\lambda^k} \Big(\overbrace{\int_{-\infty}^{+\infty} e^{-\lambda x^2} dx}^{= \sqrt{\pi/\lambda}} \Big) =$

$$= \cancel{(-1)^k} \cdot \sqrt{\pi} \cdot \cancel{(-1)^k} \cdot \underbrace{\frac{1}{2} \cdot \left(\frac{1}{2}+1\right) \cdot \ldots \cdot \left(\frac{1}{2}+k-1\right)}_{= \frac{1}{2^k} \cdot 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-3) \cdot (2k-1)} \cdot \left(\frac{2}{h}\right)^{k+1/2}$$

$$= \sqrt{\frac{2\pi}{h}} \cdot \sum_{k=0}^{\infty} (-1)^k \cdot t^{2k} \cdot \left(\frac{\cancel{2}}{h}\right)^k \cdot \frac{1}{\cancel{2^k}} \cdot \frac{\cancel{(2k-1) \cdot (2k-3) \cdot \ldots \cdot 3 \cdot 1}}{\underbrace{(2k) \cdot \cancel{(2k-1)} \cdot (2k-2) \cdot \cancel{(2k-3)} \cdot \ldots \cdot \cancel{3} \cdot 2 \cdot 1}_{= \frac{1}{2^k} \cdot \frac{1}{k!}}} =$$

$$= \sqrt{\frac{2\pi}{h}} \cdot \sum_{k=0}^{\infty} \left(-\frac{t^2}{2h}\right)^k \cdot \frac{1}{k!} = \sqrt{\frac{2\pi}{h}} \cdot \exp\left(-t^2/2h\right)$$