Tutorial #4

№ 11.2.12

a) $x\underbrace{\frac{dy}{dx}}_{=y'} + y = 3x^3y^2$

Can solve by finding integrating factor, but faster approach is to notice useful change of variable: set $p = xy$ (i.e. $y = p/x$)

Then: $p' = y + xy' \Rightarrow xy' = p' - y$

Sub this into the eq.:

$$p' \cancel{-y} \cancel{+y} = 3\left(p/x\right)^2 \cdot x^3$$

$$\frac{dp}{p^2} = 3x\,dx \Rightarrow -\frac{1}{p} + C = 3x^2/2$$

$$\Rightarrow \frac{1}{xy} + 3x^2/2 = C$$

b) $y\frac{dx}{dy} + 2x = 2x^3y^2$

$$y\,dx + 2x(1 - x^2y^2)\,dy = 0$$

Multiply the eq. by $x^\alpha y^\beta$ and try to find α, β s.t. the eq. is exact.

$$\underbrace{x^{\alpha} y^{\beta+1}}_{=: M = \frac{\partial F}{\partial x}} dx + \underbrace{\left(2x^{\alpha+1} \cdot y^{\beta} - 2x^{\alpha+3} \cdot y^{\beta+2}\right)}_{=: N = \frac{\partial F}{\partial y}} dy = 0$$

$= dF(x, y)$ if :

$$\frac{\partial M}{\partial y} = (\beta+1) x^{\alpha} \cdot y^{\beta}$$

$$\left(\text{since } \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} \right)$$

$$\frac{\partial N}{\partial x} = 2(\alpha+1) x^{\alpha} y^{\beta} - 2(\alpha+3) x^{\alpha+2} \cdot y^{\beta+2}$$

$$\begin{cases} \alpha = -3 \\ 2(\alpha+1) = \beta+1 \end{cases} \Rightarrow \begin{matrix} \alpha = -3 \\ \beta = -5 \end{matrix}$$

arbitrary function

$$F(x, y) = \int M \, dx + f(y) = \text{const on } y = y(x)$$

$$F(x, y) = \frac{1}{y^4} \underbrace{\int \frac{1}{x^3} dx}_{= -\frac{1}{2x^2}} + f(y)$$

On the other hand, $\frac{\partial F}{\partial y} = N$

$$\cancel{+ \frac{2}{y^5} \cdot \frac{1}{x^2}} + f'(y) = \cancel{2 \cdot \frac{1}{x^2} \cdot \frac{1}{y^5}} - \frac{2}{y^3}$$

$$\Rightarrow f(y) = \frac{1}{y^2} \Rightarrow -\frac{1}{2x^2 y^4} + \frac{1}{y^2} = C$$

№ 11.3.10

$$y''' - 2y'' - y' + 2y = 0$$

Sub $y = e^{\lambda x}$ to get :

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

$$\lambda^2(\lambda - 2) - (\lambda - 2) = 0 \Rightarrow (\lambda-2)(\lambda^2-1) = 0$$

$$\Rightarrow (\lambda - 1)(\lambda+1)(\lambda-2) = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = 2$$

General solution is:

$$y(x) = C_1 \cdot e^{x} + C_2 \cdot e^{-x} + C_3 \cdot e^{2x}$$

№ 11.3.12

$$x^2 y'' + xy' - 4y = 0$$

Knowing $y = x^2$ is a solution, we search for general solution in the form:

$$y(x) = C(x) \cdot x^2$$

$$x^2 \cdot \left(C'' x^2 + 4x \cdot C' + \cancel{2C}\right) + x \cdot \left(C' \cdot x^2 + \cancel{2x \cdot C}\right) - \cancel{4Cx^2} = 0$$

$$C'' \cdot x^4 = -5C' \cdot x^3$$

$$\frac{d(C')}{C'} = -\frac{5}{x}\,dx \Rightarrow \log C' = \underbrace{-5\log x}_{=\log\left(\frac{1}{x^5}\right)} + \log C_0$$

$$\Rightarrow C' = \frac{C_0}{x^5} \Rightarrow C(x) = -\frac{C_0}{4x^4} + C_1 =: c_1 + \frac{c_2}{x^4}$$

№ 11.4.13 – 11.4.16

$$\ddot{x} + \omega_0^2 x = A \cdot \sin \omega t$$

$$x(t) = \underbrace{C_1 \sin \omega_0 t + C_2 \cos \omega_0 t}_{= x_{\substack{gen. \\ hom.}}(t)} + \underbrace{C_3 \cdot \sin \omega t}_{= x_p(t)}$$

(for $\forall C_1, C_2$) (for suitable C_3)

Let's find C_3: subbing into the eq, we get:

$$(\omega_0^2 - \omega^2) \cdot C_3 \cdot \sin \omega t = A \cdot \sin \omega t$$

$$\Rightarrow C_3 = \frac{A}{\omega_0^2 - \omega^2} \quad \text{providing } \omega \neq \omega_0$$

Impose $x(0) = 0$, $\dot{x}(0) = 0$ conditions and find C_1, C_2:

$$\dot{x}(t) = C_1 \cdot \omega_0 \cdot \cos \omega_0 t - C_2 \omega_0 \cdot \sin \omega_0 t + C_3 \cdot \omega \cos \omega t$$

$$\begin{cases} x(0) = 0 \\ \dot{x}(0) = 0 \end{cases} \Rightarrow \begin{cases} C_2 = 0 \\ C_1 \cdot \omega_0 + \underbrace{C_3}_{= \frac{A}{\omega_0^2 - \omega^2}} \cdot \omega = 0 \end{cases} \Rightarrow C_1 = -\frac{A\omega}{\omega_0(\omega_0^2 - \omega^2)}$$

$$x(t) = \frac{A}{\omega_0^2 - \omega^2} \left(\sin \omega t - \frac{\omega}{\omega_0} \cdot \sin \omega_0 t \right)$$

If $\omega \approx \overset{\swarrow \omega_0}{\omega_0 + \Delta\omega}$, then :

$$x(t) = \underbrace{\frac{A}{(\omega-\omega_0)(\omega+\omega_0)}}_{\approx \frac{A}{2\Delta\omega \cdot \omega_0}} \cdot \underbrace{\left(\sin\omega_0 t - \sin\omega t\right)}_{\approx 2\cos\underbrace{\frac{(\omega+\omega_0)t}{2}}_{\approx \omega_0 t} \cdot \sin\underbrace{\frac{(\omega_0-\omega)t}{2}}_{\approx -\frac{\Delta\omega t}{2}}} \approx$$

$$\approx -\frac{A}{\omega_0 \cdot \Delta\omega} \cdot \cos\omega_0 t \cdot \sin\frac{\Delta\omega t}{2} =$$

small

$$= \underbrace{\frac{A}{\omega_0 \Delta\omega} \cdot \sin\frac{\Delta\omega t}{2}}_{\text{slow change of amplitude}} \cdot \cos(\omega_0 t + \pi)$$

has large period

slow change of amplitude - amplitude-modulation

What if $\nu = \omega_0$?

$$\ddot{x} + \omega_0^2 x = A \cdot \sin \omega_0 t$$

Let's search for a particular solution in the form (according to method of variation of parameters)

$$x_p(t) = C_1(t) \cdot \sin \omega_0 t + C_2(t) \cdot \cos \omega_0 t$$

$$C_1'' \cdot \sin\omega_0 t + 2\omega_0 \cdot C_1' \cdot \cos \omega_0 t \cancel{- C_1 \cdot \omega_0^2 \cdot \sin \omega_0 t + C_1 \cdot \omega_0^2 \sin\omega_0 t} + C_2 \cdot \omega_0^2 \cdot \cos\omega_0 t =$$

$$+ C_2'' \cdot \cos\omega_0 t - 2\omega_0 C_2' \cdot \sin\omega_0 t - C_2 \omega_0^2 \cos\omega_0 t \checkmark = A \cdot \sin \omega_0 t$$

$$\begin{cases} C_1'' - 2\omega_0 C_2' = A & \Rightarrow C_1''(t) + 4\omega_0^2 \cdot C_1(t) = A + 2\omega_0 C_0 \\ 2\omega_0 C_1' + C_2'' = 0 & \Rightarrow C_2'(t) = -2\omega_0 C_1(t) + C_0 \end{cases}$$

Since we are looking only for a particular solution, choose $C_0 = -\frac{A}{2\omega_0}$ (s.t. RHS is zero

again take a particular solution

$$C_1''(t) + 4\omega_0^2 \cdot C_1(t) = 0 \Rightarrow C_1(t) = \sin(2\omega_0 t)$$

$$\Rightarrow C_2(t) = -\frac{A}{2\omega_0} t - 2\omega_0 \cdot \int C_1(t)\, dt = \cos(2\omega_0 t) - \frac{At}{2\omega_0}$$

Then: $$x_p(t) = \underbrace{\sin(2\omega_0 t) \cdot \sin(\omega_0 t) + \cos(2\omega_0 t) \cdot \cos(\omega_0 t)}_{= \cos(\omega_0 t)} -$$

$$- \frac{At}{2\omega_0} \cdot \cos(\omega_0 t) = \cos(\omega_0 t)\left(1 - \frac{At}{2\omega_0}\right)$$

General solution:

$$x(t) = \underbrace{C_3 \cdot \sin(\omega_0 t) + C_4 \cdot \cos(\omega_0 t)}_{= x_{gen.}(t)} - \underbrace{\frac{At}{2\omega_0} \cdot \cos(\omega_0 t)}_{= \tilde{x}_p(t)}$$