Practice Test #1

(1)
$$\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{=:\sigma_y} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda_y \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0$$

$$\lambda^2 - 1 = 0 \Rightarrow \lambda_y = \pm 1 \quad \text{- e-values}$$

$\lambda_y = 1$: $\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0$

$u = -iv \Rightarrow$ ε-vector: $\begin{pmatrix} -i \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\lambda_y = -1$: $\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0$

$u = iv \Rightarrow$ ε-vector: $\begin{pmatrix} i \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{=:\sigma_x} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda_x \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\lambda^2 - 1 = 0 \Rightarrow \lambda_x = \pm 1 \quad \text{- e-values}$$

$\lambda_x = 1$: $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0 \Rightarrow u = v \Rightarrow$ ε-vector: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_x = -1$: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0 \Rightarrow u = -v \Rightarrow$ ε-vector: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(2) $$\langle f, g\rangle := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\cdot g(t)\cdot e^{-t^2/2}\, dt$$

We use Gram-Schmidt orthogonalization with normalization.

$$v_0 = C_0 \quad \Rightarrow \quad \langle v_0, v_0\rangle = \frac{C^2}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{+\infty} e^{-t^2/2}\, dt}_{=\sqrt{2\pi}} = 1$$

$$\Rightarrow \quad C_0 = 1$$

$$u_0 := v_0 = 1$$

$$v_1 = C_1 \cdot x$$

$$u_1 := C_1 x - C_1 \frac{\overbrace{\langle x, 1\rangle}^{=0 \text{ - as an integral of odd function on symmetric limits}}}{\langle 1, 1\rangle} \cdot 1$$

$$\langle u_1, u_1\rangle = 1 = \frac{C_1^2}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} t^2 e^{-t^2/2}\, dt \Rightarrow C_1 = 1$$

$$v_2 = C_2 \cdot x^2$$

$$u_2 := C_2 x^2 - C_2 \frac{\overbrace{\langle x^2, x\rangle}^{=0 \text{ - as before}}}{\langle x, x\rangle} \cdot x - C_2 \overbrace{\frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle}}^{=1}$$

$$\langle u_2, u_2\rangle = 1 = \frac{C_2^2}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} \underbrace{(t^2-1)^2}_{=t^4-2t^2+1} e^{-t^2/2}\, dt =$$

$$= \frac{C_2^2}{\sqrt{2\pi}} \cdot \sqrt{2\pi}\,(3 - 2 + 1) \quad \Rightarrow \quad C_2 = \frac{1}{\sqrt{2}}$$

$$u_0 = 1, \quad u_1 = x, \quad u_2 = \frac{1}{\sqrt{2}}(x^2 - 1)$$

Note: here we ~~all~~ the time use formula:

$$\int_{-\infty}^{+\infty} t^{2k} e^{-t^2/2}\, dt = \sqrt{2\pi} \cdot 1 \cdot 3 \cdot 5 \cdot \ldots (2k-3)\cdot(2k-1)$$

(see aside from Tutorial #3 notes)

(3) $$\begin{cases} m\ddot{x}_1 + kx_2 = 0 \\ m\ddot{x}_2 + kx_1 = 0 \end{cases}$$

Let
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot e^{i\omega t}$$

Then:
$$\begin{pmatrix} -m\omega^2 & 0 \\ 0 & -m\omega^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & -k \\ -k & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

That is:
$$\begin{pmatrix} 0 & k/m \\ k/m & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \omega^2 \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

e-value problem with $\lambda := \omega^2$

$$\omega^4 - (k/m)^2 = 0 \Rightarrow \omega^2 = \pm k/m$$

$\underline{\omega^2 = k/m}$: $$\begin{pmatrix} -k/m & k/m \\ k/m & -k/m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

$$\Rightarrow u_1 = u_2 \Rightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \underbrace{\left(C_1 \cdot \cos(\sqrt{k/m}\, t) + C_2 \cdot \sin(\sqrt{k/m} \dots \right)}_{= \tilde{c}_1 \cdot e^{i\sqrt{k/m}\, t} + \tilde{c}_2 \cdot e^{-i\sqrt{k/m} \dots}}$$

$\underline{\omega^2 = -k/m}$: $$\begin{pmatrix} k/m & k/m \\ k/m & k/m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

$$\Rightarrow u_1 = -u_2 \Rightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \left(C_3 e^{\sqrt{k/m}\, t} + C_4 \cdot e^{-\sqrt{k/m}\, t} \right)$$

Hence, there're 2 modes: oscillatory and non-oscillatory where the motion is reinforcing / dying out.

General solution is their superposition:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 \cos\left(\sqrt{k/m}\, t\right) + c_2 \cdot \sin\left(\sqrt{k/m}\, t\right) + c_3 e^{\sqrt{k/m}\, t} + c_4 e^{-\sqrt{k/m}\, t} \\ c_1 \cos\left(\sqrt{k/m}\, t\right) + c_2 \cdot \sin\left(\sqrt{k/m}\, t\right) - c_3 e^{\sqrt{k/m}\, t} - c_4 e^{-\sqrt{k/m}\, t} \end{pmatrix}$$

Can compute:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \sqrt{\frac{k}{m}} \cdot \begin{pmatrix} -c_1 \sin\left(\sqrt{k/m}\, t\right) + c_2 \cos\left(\sqrt{k/m}\, t\right) + c_3 e^{\sqrt{k/m}\, t} - c_4 e^{-\sqrt{k/m}\, t} \\ -c_1 \sin\left(\sqrt{k/m}\, t\right) + c_2 \cos\left(\sqrt{k/m}\, t\right) - c_3 e^{\sqrt{k/m}\, t} + c_4 e^{-\sqrt{k/m}\, t} \end{pmatrix}$$

Then imposing initial conditions, we have:

$$\left.\begin{aligned} \dot{x}_1(0) = 0 \quad &\Rightarrow \quad c_2 + c_3 - c_4 = 0 \\ \dot{x}_2(0) = 0 \quad &\Rightarrow \quad c_2 - c_3 + c_4 = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} c_2 &= 0 \\ c_3 &= c_4 \end{aligned}$$

$$\underbrace{\begin{aligned} x_1(0) = 2 \quad &\Rightarrow \quad c_1 + c_3 + c_4 = 2 \quad \Rightarrow \quad c_1 = 1 \\ x_2(0) = 0 \quad &\Rightarrow \quad c_1 - c_3 - c_4 = 0 \quad \Rightarrow \quad c_1 = 2c_4 \end{aligned}}$$

$$\Longrightarrow \quad c_1 = 1, \quad c_2 = 0, \quad c_3 = c_4 = 1/2$$

Finally:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \cos\left(\sqrt{k/m}\, t\right) + \cosh\left(\sqrt{k/m}\, t\right) \\ \cos\left(\sqrt{k/m}\, t\right) - \cosh\left(\sqrt{k/m}\, t\right) \end{pmatrix}$$

– note that these are exp-ly growing sol. ⇒ unphysical model

(4) $y''' + 3y'' + 3y' + y = 30 \cdot e^{-x}$

Sub $y = e^{\lambda x}$ in homogeneous eq.:

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0 \Rightarrow (\lambda+1)^3 = 0 \Rightarrow \lambda = -1$$
triple root

$\Rightarrow y''' + 3y'' + 3y' + y = 0$ has general solution:

$$y_{hom.}(x) = (C_1 + C_2 x + C_3 x^2) e^{-x}$$

Let's search for particular solution of inhomogeneous eq. in form $y_p(x) = C \cdot x^3 \cdot e^{-x} = \underbrace{x \cdot (C \cdot x^2 \cdot e^{-x})}$ writing it in this form slightly reduces further calculations

Compute: $y_p' = C \cdot 3x^2 \cdot e^{-x} + x \cdot (C x^2 e^{-x})'$

$$y_p'' = 2(C x e^{2-x})' + x \cdot (C x^2 e^{-x})''$$

$$y_p''' = 3(C x^2 e^{-x})'' + x \cdot (C \cdot x^2 e^{-x})'''$$

$$3\underbrace{(Cx^2e^{-x})''}_{= C\cdot(2 - 4x + x^2)e^{-x}} + 6\underbrace{(Cx^2e^{-x})'}_{= C\cdot(2x - x^2)e^{-x}} + 3Cx^2e^{-x} +$$

$$+ x \cdot \underbrace{\left[(Cx^2e^{-x})''' + 3(Cx^2e^{-x})'' + 3(Cx^2e^{-x})' + Cx^2e^{-x} \right]}_{=0}$$
since Cx^2e^{-x} solves homogen. eq.

$= 30 \cdot e^{-x}$, that is

$$C \cdot (6 \cancel{- 12x} \cancel{+ 3x^2} \cancel{+ 12x} \cancel{- 6x^2} \cancel{+ 3x^2}) e^{-x} = 30 \cdot e^{-x}$$

$\Rightarrow C = 5 \Rightarrow y(x) = (C_1 + C_2 x + C_3 x^2) e^{-x} + 5x^3 e^{-x}$

5

$$\begin{cases} \dot{x}_1 = 2x_1 + x_2 \\ \dot{x}_2 = x_1 - x_2 \end{cases}, \qquad \begin{matrix} x_1(0) = 1 \\ x_2(0) = 2 \end{matrix}$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$

$$-(2-\lambda)(1+\lambda) - 1 = 0$$

$$\lambda^2 - \lambda - 3 = 0 \implies \lambda_{1,2} = \frac{1 \pm \sqrt{13}}{2}$$

E-vectors: $\begin{pmatrix} 1 \\ \frac{-3+\sqrt{13}}{2} \end{pmatrix}$, $\begin{pmatrix} 1 \\ \frac{-3-\sqrt{13}}{2} \end{pmatrix}$

$$\implies S = \begin{pmatrix} 1 & 1 \\ \frac{-3+\sqrt{13}}{2} & \frac{-3-\sqrt{13}}{2} \end{pmatrix}, \quad D = \begin{pmatrix} \frac{1+\sqrt{13}}{2} & 0 \\ 0 & \frac{1-\sqrt{13}}{2} \end{pmatrix}$$

$$AS = SD \implies A = SDS^{-1}$$

$$e^{At} = S \begin{pmatrix} e^{\frac{1+\sqrt{13}}{2}t} & 0 \\ 0 & e^{\frac{1-\sqrt{13}}{2}t} \end{pmatrix} S^{-1} = \ldots$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{At} \cdot \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \ldots = \begin{pmatrix} \frac{1}{26}\sqrt{13}\left[(7+\sqrt{13})e^{\frac{1+\sqrt{13}}{2}t} - (7-\sqrt{13})e^{\frac{1-\sqrt{13}}{2}t}\right] \\ \left(1-\frac{2}{\sqrt{13}}\right)e^{\frac{1+\sqrt{13}}{2}t} + \left(1+\frac{2}{\sqrt{13}}\right)e^{\frac{1-\sqrt{13}}{2}t} \end{pmatrix}$$

(6)

1) This is true only for normal matrices (special class of matrices that can be diagonalized by unitary transform).
But, in general, false. Take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$A A^T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

2) $A \in \mathbb{R}^{5\times 5} \Rightarrow$ e-values are roots of a polynomial of degree 5 with real coefficients

$\Downarrow$ there're 5 roots (e-values)

$\Downarrow$ if complex, they necessarily come in conjugated pairs

$\Rightarrow$ there's at least 1 real e-value

3) $\exists A^{-1} \Leftrightarrow \det A \neq 0 \Leftrightarrow \underbrace{(\det A)^2}_{= \det A \,\cdot\, \det A} \neq 0 \Leftrightarrow$

$\det A = \det A^T$

$\Leftrightarrow \det(A^T \cdot A) \neq 0 \Leftrightarrow \exists (A^TA)^{-1}$

4) Let λ be an e-value of A with e-vector v.

Consider

$$e^{At}\cdot v = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} v = \sum_{k=0}^{\infty} \overbrace{A^k v}^{=\lambda^k v} \cdot \frac{t^k}{k!} =$$

$$= \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k t^k}{k!}}_{= e^{\lambda t}} v = e^{\lambda t} v \implies e^{\lambda t} \text{ is an e-value of } e^{At}$$

5) $S = S^T \not\Rightarrow \det(e^S) = e^{\det S}$

Take $S = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \implies \det S = 0$

$$\implies e^S = \begin{pmatrix} e^2 & 0 \\ 0 & 1 \end{pmatrix}$$

$\implies \det(e^S) = e^2 \neq 1 = e^{\det S}$

6) Let $B := A + A^T$. Note: $\underset{A+A^T}{B} = \underset{= A^T + A}{B^T}$

B - symmetric $\implies \underset{\text{non-singular}}{\exists S} : B = S \underset{\text{diagonal with possibly zero elements}}{D} S^{-1}$

$\implies e^B = S \underset{\text{diagonal with no zero elements}}{e^D} S^{-1}$

$$\det e^B = \cancel{\det(S)} \cdot \det(e^D) \cdot \frac{1}{\cancel{\det(S)}} = \det(e^D) \neq 0$$

7) H - Hermitian matrix $\Rightarrow$ all its e-values are real (See the bottom of the page).

Let λ be an e-value for e-vector v.

Consider

$$e^{iH} v = \sum_{k=0}^{\infty} \frac{i^k \overbrace{H^k v}^{=\lambda^k v}}{k!} = \underbrace{\sum_{k=0}^{\infty} \frac{i^k \lambda^k}{k!}}_{= e^{i\lambda}} \cdot v$$

$\Rightarrow$ $e^{i\lambda}$ is an e-value of e^{iH} and since λ is real: $|e^{i\lambda}| = 1$

But λ was an arbitrary e-value of H, So all e-values of e^{iH} lie on the unit circle in complex plane.

Aside (to recall that $\lambda \in \mathbb{R}$):

Consider

$$\underbrace{\langle v, \underbrace{Hv}_{=\lambda v} \rangle}_{= \lambda \langle v, v \rangle} = \langle \underbrace{H^+}_{=H} v, v \rangle = \underbrace{\langle \lambda v, v \rangle}_{= \lambda^* \langle v, v \rangle}$$

$$\Rightarrow (\lambda - \lambda^*) \cdot \overbrace{\|v\|^2}^{>0} = 0 \Rightarrow \lambda = \lambda^*$$