N 12.1.9 / N 12.3.6

Tutorial # 6

$$(1-x^2)y'' - 2xy' + 2y = 0$$

Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$\underbrace{\sum_{n=0}^{\infty} a_n \cdot n(n-1) \cdot x^{n-2}}_{= \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}} - \sum_{n=0}^{\infty} a_n \cdot n(n-1) x^n -$$

$$- \sum_{n=0}^{\infty} 2a_n \cdot n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

Consider the 1st term:

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = \left\{ \begin{array}{l} k := n-2 \\ n = k+2 \end{array} \right\} = \sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) \cdot x^k =$$

$$= \left\{ \begin{array}{l} \text{Rename} \\ \text{index } k \text{ to } n \end{array} \right\} = \sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n$$

Then:

$$\sum_{n=0}^{\infty} \Big[a_{n+2} \cdot (n+2)(n+1) \underbrace{- a_n \cdot n(n-1) \overbrace{- 2a_n \cdot n + 2a_n}^{= -2a_n(n-1)}}_{= -a_n(n-1)\cdot(n+2)} \Big] x^n = 0$$

We require $[\dots] = 0$ $\Rightarrow$ $a_{n+2} = \dfrac{(n-1) \cdot \cancel{(n+2)}}{\cancel{(n+2)} \cdot (n+1)} \cdot a_n$

$n = 1: \quad a_3 = 0 \quad \Rightarrow \quad a_5 = a_7 = \dots = a_{2k+1} = \dots = 0$

(i.e. the solution will include only even powers of x)

$n = 0: \quad a_2 = -a_0$

$n = 2k$:

$$a_{2k+2} = \frac{2k-1}{2k+1} \cdot a_{2k} = \frac{2k-1}{2k+1} \cdot \frac{2k-3}{2k-1} \cdot a_{2k-2} =$$

$$= \frac{\cancel{(2k-1)}}{2k+1} \cdot \frac{\cancel{(2k-3)}}{\cancel{(2k-1)}} \cdot \frac{2k-5}{\cancel{(2k-3)}} \, a_{2k-4} = \dots = \frac{1}{2k+1} \, a_2 =$$

$$= - \frac{1}{2k+1} \, a_0$$

The solution is:

$$y(x) = a_0 + a_1 + \sum_{k=0}^{\infty} \underbrace{a_{2k+2}}_{= -\frac{a_0}{2k+1}} \cdot \overbrace{x^{2k+2}}^{x \cdot x^{2k+1}} = a_1 + a_0 \; x \underbrace{\sum_{k=0}^{\infty} \frac{a_0}{2k+1} \cdot x^{2k+1}}_{\text{can be summed}}$$

Recall:

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{k=1}^{\infty} \frac{x^{2k}}{2k} + \sum_{k=0} \frac{x^{2k+1}}{2k+1}$$

$$\ln(1-x) = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^n}{n} = \sum_{k=1}^{\infty} \frac{x^{2k}}{2k} - \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$$

$$\Longrightarrow \ln(1+x) - \ln(1-x) = 2 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$$

$$\Longrightarrow \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} = \frac{1}{2} \cdot \ln \frac{1+x}{1-x}$$

$$y(x) = a_1 x + a_0 \left(1 - \frac{x}{2} \cdot \ln \frac{1+x}{1-x}\right)$$

№ 12.1.8

$$xy' - y = x^2 e^x$$

Let $y(x) = \sum\limits_{k=0} a_k x^k$.

$$x^2 e^x = \sum_{k=0}^{\infty} \frac{x^{k+2}}{k!} = \left\{ \begin{matrix} k+2 =: n \\ k = n-2 \end{matrix} \right\} = \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!} ;$$

Subbing these into eq. results in:

$$\underbrace{\sum_{k=0}^{\infty} a_k k x^k}_{= \sum\limits_{k=1}^{\infty} a_k k x^k} - \sum_{k=0}^{\infty} a_k x^k = \sum_{k=2}^{\infty} \frac{x^k}{(k-2)!}$$

Now match the coefficients at different powers of x in both sides:

$k=0$: $0 - a_0 = 0 \Rightarrow a_0 = 0$

$k=1$: $a_1 - a_1 = 0 \Rightarrow a_1$ is arbitrary!

$k \geq 2$: $a_k \cdot (k-1) = \dfrac{1}{(k-2)!} \Rightarrow a_k = \dfrac{1}{(k-1)!}$

Solution:

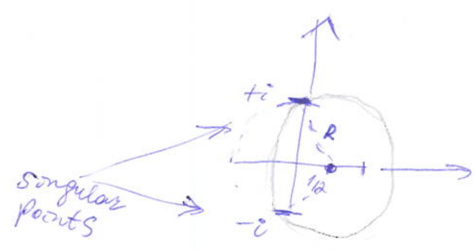
$$y(x) = a_1 x + \underbrace{\sum_{k=2} \frac{x^k}{(k-1)!}}_{= x\sum\limits_{k=2}^{\infty} \frac{x^{k-1}}{(k-1)!} = x \cdot \sum\limits_{n=1}^{\infty} \frac{x^n}{n!} = x(e^x - 1)} = \overset{= a_1 - 1}{C} \cdot x - x e^x$$

C – arbitrary const

№ 12.2.7

$$(1+x^2) y''(x) + x y'(x) + 6x^2 y(x) = 0$$

$$p(x) = \frac{x}{1+x^2}, \quad q(x) = \frac{6x^2}{1+x^2}$$



When expanded about $x = 1/2$, radius of convergence is $R = \sqrt{1 + (1/2)^2} = \frac{\sqrt{5}}{2}$

~ 12. 3. 10 – 12. 3. 11

Legendre's eq.:

$$\underbrace{(1-x^2)y'' - 2xy'}_{=\left[(1-x^2)y'\right]'} + \alpha(\alpha+1)y = 0$$

Let $\alpha = n$ / non-negative integers. Denote solution for such equation as $P_n(x)$.

$\alpha = m$ —— " —— $P_m(x)$.

That is, $P_n(x)$, $P_m(x)$ satisfy:

$$\left[(1-x^2)P_n'(x)\right]' + n(n+1)\cdot P_n(x) = 0 \quad (1)$$

$$\left[(1-x^2)P_m'(x)\right]' + m(m+1)\cdot P_m(x) = 0 \quad (2)$$

Consider:

$$\int_{-1}^{1} P_m(x)\cdot (1)\, dx - \int_{-1}^{1} P_n(x)\cdot (2)\, dx = 0$$

$$\underbrace{\int_{-1}^{1} P_m(x)\cdot\left[(1-x^2)\cdot P_n'(x)\right]' dx}_{=\underbrace{P_m(x)\cdot P_n(x)\cdot(1-x^2)\Big|_{-1}^{1}}_{=0} - \int_{-1}^{1} P_m'(x)\cdot P_n'(x)\cdot(1-x^2)dx} = \underbrace{\int_{-1}^{1} P_n(x)\left[(1-x^2)P_m'(x)\right]' dx}_{=0-\int_{-1}^{1} P_n'(x)\cdot P_m'(x)(1-x^2)dx} = \left(m(m+1)-n(n+1)\right)\cdot\int_{-1}^{1} P_n(x)\cdot P_m(x)dx$$

$$\Rightarrow \overbrace{\left[m(m+1)-n(n+1)\right]}^{=m^2-n^2+m-n=(m-n)(m+n+1)}\cdot\int_{-1}^{1} P_n(x)\cdot P_m(x)\,dx = 0 \Rightarrow \int_{-1}^{1} P_n(x)\cdot P_m(x)\,dx = 0$$

for $m \neq n$