Tutorial # 7

N 12.4.7

$$4x^2y'' - 2x(x+2)y' + (x+3)y = 0$$

let $y(x) = x^r \cdot \sum_{n=0}^{\infty} a_n \cdot x^n$ (since $x=0$ is a regular singular point

$$y'(x) = x^r \sum_{n=0}^{\infty} (n+r)\, a_n\, x^{n-1}, \quad y''(x) = x^r \sum_{n=0}^{\infty} (n+r)(n+r-1) \cdot a_n x^n$$

$$\sum_{n=0}^{\infty} \Big[\overbrace{4a_n(n+r)(n+r-1) - 4a_n(n+r)}^{=4a_n(n+r)(n+r-2)} + 3a_n\Big] x^{n+r} -$$

$$- \underbrace{\sum_{n=0}^{\infty} \big[2(n+r)\, a_n - a_n\big] x^{n+r+1}}_{= \sum_{k=1}^{\infty} \left[2(k-1+r)\, a_{k-1} - a_{k-1}\right] x^{k+r} \;=\; \sum_{n=1}^{\infty} 2(n-\frac{3}{2}+r)\, a_{n-1} x^n} = 0$$

At lowest power of x ($n=0$ in 1st series) : $4r(r-2) + 3 = 0$

$$r_{1,2} = 1 \pm 1/2$$

$$a_n = \frac{2(n+r) - 3}{4(n+r)(n+r-2) + 3}\, a_{n-1}$$

$r = 3/2$: $a_n = \dfrac{2n}{(2n+3)(2n-1)+3}\, a_{n-1} = \dfrac{a_{n-1}}{2(n+1)} = \dfrac{a_0}{2^n (n+1)!}$

$r = 1/2$: $a_n = \dfrac{2(n-1)}{(2n+1)(2n-3)+3} = \dfrac{a_{n-1}}{2n} = \dfrac{a_0}{2^n \cdot n!}$

$$y(x) = C_1 \underbrace{x^{3/2} \cdot \sum_{n=0}^{\infty} \frac{x^n}{2^n (n+1)!}}_{= x^{1/2} \cdot \left(2 \sum_{n=0}^{\infty} \frac{x^n}{2^n \cdot n!} - 2\right)} + C_2 \cdot x^{1/2} \cdot \sum_{n=0}^{\infty} \frac{x^n}{2^n n!} = \tilde{C}_1 \cdot x^{1/2} + \tilde{C}_2 \cdot x^{1/2} \cdot \sum \frac{x^n}{2^n \cdot n!}$$

№ 12.5.16

$$y''(x) + a^2 x \cdot y(x) = 0$$

We want to do change of variables rescaling x and y.
Start with argument change:

Let $z := c \cdot x^{\beta}$. Then: $y' = \frac{\partial y}{\partial x} = \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x}$

$\Rightarrow x = (z/c)^{1/\beta}$ $\qquad y'' = \frac{\partial^2 y}{\partial z^2} \cdot \left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial y}{\partial z} \cdot \frac{\partial^2 z}{\partial x^2}$

Compute $\frac{\partial z}{\partial x} = \beta \cdot c \cdot x^{\beta - 1}$

$$\frac{\partial^2 z}{\partial x^2} = \beta(\beta - 1) \cdot c \cdot x^{\beta - 2}$$

$$y''(x) = \underbrace{\frac{\partial^2 y}{\partial z^2}}_{= y''(z)} \cdot \beta^2 c^2 \cdot x^{2\beta - 2} + \underbrace{\frac{\partial y}{\partial z}}_{= y'(z)} \cdot \beta(\beta - 1) \cdot c \cdot x^{\beta - 2}$$

Then the eq. becomes:

$$\beta^2 c^2 x^{2\beta-2} \cdot y''(z) + c x^{\beta-2} \beta \cdot (\beta - 1) \cdot y'(z) + a^2 \cdot x \cdot y(z) = 0$$

$$\beta^2 \cdot c^{\cancel{2} - \frac{2\beta-2}{\beta}} \cdot z^{\frac{2\beta-2}{\beta}} \cdot y''(z) + \beta(\beta-1) c^{\cancel{1} - \frac{\beta-2}{\beta}} \cdot z^{\frac{\beta-2}{\beta}} y'(z) + a^2 \cdot c^{-1/\beta} \cdot z^{1/\beta} \cdot y(z) = 0$$

Divide by $c^{-\frac{2}{\beta}} \cdot z^{-\frac{2}{\beta}}$:

$$\beta^2 \cdot z^2 \cdot y''(z) + \beta(\beta - 1) \cdot z \cdot y'(z) + a^2 \cdot c^{\frac{1}{\beta}} \cdot z^{3/\beta} y(z) = 0$$

Now introduce $\rho(z)$ s.t. $y(z) = z^{\alpha}\cdot\rho(z)$

Then:

$$y'(z) = \alpha z^{\alpha-1}\cdot\rho(z) + z^{\alpha}\cdot\rho'(z)$$

$$y''(z) = \alpha(\alpha-1)\cdot z^{\alpha-2}\cdot\rho(z) + 2\alpha z^{\alpha-1}\rho'(z) + z^{\alpha}\cdot\rho''(z)$$

The eq. finally becomes:

$$\beta^2 z^{\alpha+2}\rho''(z) + \left(\beta(\beta-1)+2\alpha\beta^2\right)z^{\alpha+1}\cdot\rho'(z) + \left[\left(\beta^2\alpha(\alpha-1)+\beta(\beta-1)\alpha\right)z^{\alpha} + a^2 c^{\frac{1}{\beta}} z^{\alpha+\frac{3}{\beta}}\right]\rho(z) = 0$$

Divide by $\beta^2 z^{\alpha}$:

$$z^2\cdot\rho''(z) + \underbrace{\frac{\beta-1+2\alpha\beta}{\beta}}_{=1}\cdot z\rho'(z) + \left[\overbrace{\frac{\beta\alpha(\alpha-1)+(\beta-1)\alpha}{\beta}}^{=\frac{\alpha(2\beta-1)}{\beta}} + \underbrace{\frac{a^2}{\beta^2}c^{\frac{1}{\beta}}z^{\frac{3}{\beta}}}_{=z^2}\right]\rho(z) = 0$$

In order to have Bessel's eq., we require:

$$\begin{cases} \frac{a^2}{\beta^2}\cdot c^{1/\beta}\cdot z^{3/\beta} = z^2 & \Rightarrow \quad \beta = 3/2,\quad c^{2/3} = \frac{9}{4a^2} \Rightarrow c = \left(\frac{3}{2a}\right)^3 \\ \frac{\beta-1+2\alpha\beta}{\beta} = 1 & \Rightarrow \quad \alpha = \frac{1}{2\beta} = \frac{1}{3} \end{cases}$$

Then:

$$z^2\cdot\rho''(z) + z\cdot\rho'(z) + \left(z^2 - \underbrace{\frac{1}{9}}_{(\nu)^2}\right)\cdot\rho(z) = 0$$

— Bessel's eq. with $\nu = \frac{1}{3}$

Bessel's functions

$$\Rightarrow \rho(z) = C_1\cdot J_{1/3}(z) + C_2\cdot Y_{1/3}(z)$$

— general solution.

Getting back to the original variables, we obtain:

$$y(x) = \left[\left(\frac{3}{2a}\right)^3\cdot x^{3/2}\right]^{1/3}\cdot\rho\left(\left(\frac{3}{2a}\right)^3\cdot x^{3/2}\right) = \tilde{C}_1\cdot x^{1/2}\cdot J_{1/3}\left(\left(\frac{3}{2a}\right)^3\cdot x^{3/2}\right) + \tilde{C}_2\cdot x^{1/2}\cdot Y_{1/3}\left(\left(\frac{3}{2a}\right)^3\cdot x^{3/2}\right)$$

~ 12.5.17

$$y''(x) + p(x)\,y'(x) + q(x)\,y(x) = 0 \qquad (*)$$

Let $y_1(x)$, $y_2(x)$ be 2 lin. indep. sol. to $(*)$:

$$y_1'' + p \cdot y_1' + q \cdot y_1 = 0 \qquad (1)$$

$$y_2'' + p \cdot y_2' + q \cdot y_2 = 0 \qquad (2)$$

Note that combination $y_2 \cdot (1) - y_1 \cdot (2)$ eliminates the 3^{rd} terms:

$$\underbrace{y_2 \cdot y_1'' - y_1 \cdot y_2''}_{= \left(y_2 \cdot y_1' - y_1 \cdot y_2'\right)' = -W'[y_1, y_2]} + p \cdot \underbrace{\left(y_2 \cdot y_1' - y_1 \cdot y_2'\right)}_{= -W[y_1, y_2]} = 0$$

$$W'[y_1, y_2] + p \cdot W[y_1, y_2] = 0$$

$$\Rightarrow \quad W[y_1, y_2] = A \cdot e^{-\int p(x)\,dx}$$

If $(*)$ is Bessel's eq., then $p(x) = \frac{x}{x^2} = \frac{1}{x}$

$$W[y_1, y_2] = A \cdot e^{-\ln x} = \frac{A}{x}$$

N 12.6.10

Need to show: $e^{ix\sin\theta} = \sum_{n=-\infty}^{+\infty} e^{in\theta} \cdot J_n(x)$

Take $t = e^{i\theta}$ in the generating function:

$$e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{+\infty} J_n(x) \cdot t^n$$

$$e^{\frac{x}{2}\left(e^{i\theta} - e^{-i\theta}\right)} = \sum_{n=-\infty}^{+\infty} J_n(x)\, e^{in\theta}$$

$$e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\theta}$$