Tutorial #8

~ 14. 1. 15

$$I = \int_{-1}^{1} x \cdot P_n(x) \cdot P_m(x)$$

Recall the recursion formula:

$$(n+1) P_{n+1}(x) - (2n+1)\, x P_n(x) + n P_{n-1}(x) = 0 \qquad n \geq 1$$

$$\Rightarrow x P_n(x) = \frac{n+1}{2n+1} \cdot P_{n+1}(x) + \frac{n}{2n+1} P_{n-1}(x)$$

By orthogonality we have:

$$\int_{-1}^{1} P_m(x)\, P_k(x)\, dx = \frac{2}{2k+1} \delta_{mk}$$

Then

$$I = \frac{n+1}{2n+1} \underbrace{\int_{-1}^{1} P_m(x)\, P_{n+1}(x)\, dx}_{\frac{2}{2n+3}\delta_{m,n+1}} + \frac{n}{2n+1} \underbrace{\int_{-1}^{1} P_m(x)\, P_{n-1}(x)\, dx}_{\frac{2}{2n-1}\delta_{m,n-1}} =$$

$$= \frac{2(n+1)}{(2n+1)(2n+3)} \delta_{m,n+1} + \frac{2n}{(2n+1)(2n-1)} \delta_{m,n-1}$$

N 14.2.6

$$G(x,t) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)\, t^n$$

$\hookrightarrow$?

Recall: $H_{n+1}(x) - 2x H_n(x) + 2n H_{n-1}(x) = 0 \qquad (*)$

$\sum_{n=1}^{\infty} (*) \cdot \frac{t^n}{n!}$:

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \cdot H_{n+1}(x) = \overbrace{\sum_{n=1}^{\infty} \frac{t^n}{n!} \cdot H_{n+1}(x)}^{= \sum_{k=2}^{\infty} \frac{t^{k-1}}{(k-1)!} H_k(x)} =$$

$$= \sum_{k=2}^{\infty} \frac{k \cdot t^{k-1}}{k!} H_k(x) = \frac{\partial G}{\partial t}(x,t) - H_1(x)$$

$$\sum_{n=1}^{\infty} 2x \frac{t^n}{n!} H_n(x) = 2x \left(G(x,t) - H_0(x) \right)$$

$$\sum_{n=1}^{\infty} \frac{2n\, t^n}{n!} \cdot H_{n-1}(x) = 2t \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} H_{n-1}(x) =$$

$$= 2t \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x) = 2t\, G(x,t)$$

Putting this together gives:

$$\frac{\partial G(x,t)}{\partial t} + 2(t-x) G(x,t) + \underbrace{2x \overset{=1}{H_0(x)} - \overset{=2x}{H_1(x)}}_{=0} = 0 \qquad \text{- ODE for fixed } x$$

$$\frac{\partial G}{\partial t} + 2(t - x) G = 0 \Rightarrow G(x,t) = C(x) \cdot e^{-t^2 + 2tx}$$

$$G(x, 0) = C(x) = H_0(x) = 1$$

$$\Rightarrow G(x,t) = e^{2tx - t^2}$$

N 14. 3. 9

$$\begin{cases} y'' + \lambda^2 y = 0 \\ y'(0) = 0, \quad y'(1) = 0 \end{cases}$$

$$y(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x), \quad \lambda \geq 0$$

$$y'(x) = -C_1 \lambda \sin(\lambda x) + C_2 \lambda \cos(\lambda x)$$

$$y'(0) = 0 \Rightarrow C_2 = 0$$

$$y'(1) = 0 \Rightarrow C_1 \lambda \cdot \sin(\lambda) = 0 \Rightarrow \lambda_n = \pi n$$

$$y_n(x) = \cos(\pi n x), \quad n = 0, 1, \ldots$$

№ 14.3.7 / № 14.4.3

$$\begin{cases} y'' + \lambda^2 y = 0 \\ y(0) = 0, \quad y'(l) = 0 \end{cases}$$

$$y(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x), \quad \lambda \geq 0$$

(w.l.o.g. since C_1, C_2 are arbitrary)

$$y'(x) = -C_1 \lambda \sin(\lambda x) + C_2 \lambda \cos(\lambda x)$$

$$y(0) = 0 \Rightarrow C_1 = 0$$

$$y'(l) = 0 \Rightarrow C_2 \lambda \cos(\lambda l) = 0 \Rightarrow \lambda_n l = \frac{\pi}{2} + \pi n$$

$$\Rightarrow \lambda_n = \frac{\pi}{2l}(1+2n), \quad n = 0, 1, \ldots$$

$$y_n(x) = \sin\left(\frac{\pi}{2l}(1+2n)x\right)$$

Let $l = 1$. Then: $y_n(x) = \sin\left(\frac{\pi}{2}(1+2n) \cdot x\right)$

We want to expand $f(x) := x \cdot (1-x)^2$ in $y_n(x)$'s

$$x \cdot (1-x)^2 = \sum_{n=0}^{\infty} c_n \cdot \sin\left(\frac{\pi}{2}(1+2n)x\right) \qquad (*)$$

c_n ?

Fix $m \geq 0$ and do $\int_0^1 (*) \cdot \sin\left(\frac{\pi}{2}(1+2m)x\right) dx$:

$$\int_0^1 \sin\left(\frac{\pi}{2}(1+2m)x\right) \cdot x \cdot (1-x)^2 \, dx = \sum_{n=0}^{\infty} c_n \cdot \int_0^1 \underbrace{\sin\left(\frac{\pi}{2}(1+2n)x\right) \cdot \sin\left(\frac{\pi}{2}(1+2m)x\right)}_{\frac{1}{2}\left[\cos(\pi(n-m)x) - \underbrace{\cos\left(\pi(1+(n+m))x\right)}_{= -\cos(\pi(n+m)x)}\right]} dx$$

$$\int_0^1 \sin\left(\frac{\pi}{2}(1+2n)x\right) \sin\left(\frac{\pi}{2}(1+2m)x\right) dx =$$

$$= \frac{1}{2}\int_0^1 \cos\left(\pi(n-m)x\right) dx + \frac{1}{2}\int_0^1 \cos\left(\pi(n+m)x\right) dx =$$

$$= 0$$

only for $n \neq m$

$$\frac{1}{2}\delta_{nm}$$

$$\Rightarrow \quad C_m = \int_0^1 \sin\left[\frac{\pi}{2}(1+2m)x\right] \cdot x(1-x)^2 \, dx =$$

integrate 3 times by parts ...