Practice test #2

№ 1

$$\begin{cases} \dfrac{dS}{dt} = -\beta I S + \gamma R \\ \dfrac{dI}{dt} = \beta S I - \alpha I \\ \dfrac{dR}{dt} = \alpha I - \gamma R \end{cases}, \qquad \begin{array}{l} \alpha, \beta, \gamma > 0 \\ S, I, R \geq 0 \\ \underbrace{S + I + R = 1} \end{array}$$

invariant region since $\dfrac{d(S+I+R)}{dt} = -\beta IS + \gamma R + \beta SI - \alpha I + \alpha I - \gamma R = 0$

$S + I + R = 1 \implies R = 1 - S - I$ this constraint reduces the dimension of the system:

$$\begin{cases} \dfrac{dS}{dt} = -\beta I S - \gamma S - \gamma I + \gamma \\ \dfrac{dI}{dt} = \beta S I - \alpha I \end{cases}$$

Linearization about (S_0, I_0) pt.:

$$\frac{d}{dt}\begin{pmatrix} S \\ I \end{pmatrix} = \underbrace{\begin{pmatrix} -\beta I_0 - \gamma & -\beta S_0 - \gamma \\ \beta I_0 & \beta S_0 - \alpha \end{pmatrix}}_{=: J} \begin{pmatrix} S - S_0 \\ I - I_0 \end{pmatrix}$$

Let's find critical pts.:

$$\begin{cases} \beta I S + \gamma S + \gamma I = \gamma \\ I \cdot (\beta S - \alpha) = 0 \end{cases}$$

① $I_0 = 0 \implies S_0 = 1 \implies \boxed{(S_0, I_0) = (1, 0) \quad R_0 = 0}$

② $S_0 = \alpha/\beta \implies I_0 = \dfrac{\gamma(1 - \alpha/\beta)}{\alpha + \gamma} \implies \boxed{(S_0, I_0) = \left(\alpha/\beta, \dfrac{\gamma}{\beta} \cdot \dfrac{\beta - \alpha}{\gamma + \alpha}\right), \quad R_0 = 1 - \dfrac{\alpha^2 + \beta\gamma}{\beta(\gamma + \alpha)} = \dfrac{\alpha}{\beta} \cdot \dfrac{\beta - \alpha}{\gamma + \alpha}}$

Note: since $I_0 > 0$ we must have $\alpha < \beta$.

(1) $J = \begin{pmatrix} -\gamma & -\beta-\gamma \\ 0 & \beta-\alpha \end{pmatrix} \Rightarrow \begin{array}{l} \lambda_1 = -\gamma < 0 \\ \lambda_2 = \beta - \alpha \end{array}$

$\Rightarrow (S_0, I_0, R_0) = (1, 0, 0)$ is: $\begin{cases} \text{a stable node (may be proper or improper if } \alpha = \gamma \text{),} & \text{if } \beta < \alpha; \\ \text{a saddle node} & \text{if } \beta > \alpha. \end{cases}$

(2) $J = \begin{pmatrix} -\gamma \cdot \underbrace{\left(\frac{\beta-\alpha}{\gamma+\alpha} + 1\right)}_{= \frac{\gamma+\beta}{\gamma+\alpha}} & -\alpha-\gamma \\ \gamma \cdot \frac{\beta-\alpha}{\gamma+\alpha} & 0 \end{pmatrix}$

Shortcut:

$\begin{cases} \det J = \lambda_1 \cdot \lambda_2 = \alpha \cdot 0 + \gamma \overbrace{(\alpha+\gamma)}^{>0} \cdot \overbrace{\frac{\beta-\alpha}{\gamma+\alpha}}^{>0 \text{ since } \alpha > \beta} > 0 \\ \operatorname{Tr} J = \lambda_1 + \lambda_2 = -\gamma \cdot \underbrace{\frac{\gamma+\beta}{\gamma+\alpha}}_{>0} - \alpha < 0 \end{cases}$

$\Rightarrow \lambda_1, \lambda_2 < 0$

$\Rightarrow (S_0, I_0, R_0) = \left(\alpha/\beta, \ \frac{\gamma}{\beta} \cdot \frac{\beta-\alpha}{\gamma+\alpha}, \ \frac{\alpha}{\beta} \cdot \frac{\beta-\alpha}{\gamma+\alpha} \right)$

is a stable node.

№ 2

Idea is the same as in the problem № 14.1.15 covered last time in the tutorial.

№3

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k!)^2} \cdot x^{2k}$$

$$J_0'(x) = \sum_{k=1}^{\infty} \frac{2k\cdot(-1)^k}{2^{2k}(k!)^2} \cdot x^{2k-1} =$$

$$J_0''(x) = \sum_{k=1}^{\infty} \frac{2k(2k-1)(-1)^k}{2^{2k}\cdot(k!)^2} \cdot x^{2k-2}$$

$$J_0''(x) + \frac{1}{x}J_0'(x) + J_0(x) = \sum_{k=1}^{\infty} \frac{2k(2k-1)(-1)^k}{2^{2k}(k!)^2} x^{2k-2} +$$

$$+ \sum_{k=1}^{\infty} \frac{2k\cdot(-1)^k}{2^{2k}(k!)^2} x^{2k-2} + \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k!)^2} x^{2k}} =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{2n-2}\cdot \underbrace{[(n-1)!]^2}_{=\frac{(n!)^2}{n^2}}} \cdot x^{2n-2}$$

$k = n-1$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k}\cdot(k!)^2} \underbrace{\left[2k(2k-1) + 2k - 2^2\cdot k^2\right]}_{=0} x^{2k-2} = 0$$

№ 4

1) $f(x) = \sum_{n=0}^{\infty} a_n \cdot y_n(x), \; x \in [-l, l]$; $\int_{-l}^{l} [y_n(x)]^2 \, dx = 1$

(*) $\frac{1}{l} \cdot \int_{-l}^{l} f^2(x) \, dx \geq \sum_{n=1}^{N} a_n^2 + \frac{a_0^2}{2}$ – Bessel's inequality

2) $f(x) = |\sin x|, \quad x \in [-\pi, \pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos(nx) + b_n \cdot \sin(nx) \right)$$

Since $|\sin x|$ is even and $\sin(nx)$ are odd, we conclude that all $b_n = 0$, $n = 1, 2, \dots$.

$$a_n = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \underbrace{|\sin x| \cdot \cos(nx)}_{\text{even function}} \, dx = \frac{2}{\pi} \cdot \int_{0}^{\pi} \sin x \cdot \cos(nx) \, dx =$$

$$= \frac{2}{\pi} \cdot \frac{1}{2} \int_{0}^{\pi} \left(\sin[(n+1)x] - \sin[(n-1)x] \right) dx =$$

$$= \frac{1}{\pi} \cdot \Bigg(\underbrace{- \frac{\cos[(n+1)x]}{n+1} \Bigg|_{0}^{\pi}}_{= \frac{-(-1)^{n+1} + 1}{n+1} = \frac{(-1)^n + 1}{n+1}} - \underbrace{\int_{0}^{\pi} \sin[(n-1)x] \, dx}_{\substack{\text{may turn to zero} \Rightarrow \\ \text{need to consider} \\ \text{the case } n=1 \text{ separately}}} \Bigg), \quad n = 0, 1, \dots$$

$n = 1: \quad a_1 = 0$

$$n \neq 1: \quad a_n = \frac{1}{\pi} \cdot \left[\frac{(-1)^n + 1}{n+1} + \underbrace{\frac{\cos[(n-1)x]}{n-1} \Bigg|_{0}^{\pi}}_{= \frac{(-1)^{n-1} - 1}{n-1} = -\frac{(-1)^n + 1}{n-1}} \right] = \frac{1}{\pi} \left[(-1)^n + 1 \right] \cdot \left(\frac{1}{n+1} - \frac{1}{n-1} \right)$$

$$= -\frac{1}{\pi} \left[(-1)^n + 1 \right] \cdot \frac{2}{n^2 - 1}, \quad n = 0, 2, 3, 4, \dots$$

$$f(x) = \frac{2}{\pi} + \frac{2}{\pi} \cdot \sum_{n=2}^{\infty} \frac{\overbrace{\left[(-1)^n + 1\right]}^{\neq 0 \text{ for even } n \text{ only}}}{n^2 - 1} \cos(nx) =$$

$n = 2k$

since only terms with even n survive

$$= \frac{2}{\pi} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{2 \cdot \cos(2kx)}{4k^2 - 1} = \frac{2}{\pi} \cdot \left[1 + 2 \cdot \sum_{k=1}^{\infty} \frac{\cos(2kx)}{4k^2-1}\right]$$

3) As $N \to \infty$, Bessel's inequality (*) turns into Parseval's identity:

$$\frac{1}{\pi} \cdot \int_{-\pi}^{\pi} |\sin x|^2 \, dx = \sum_{n=1}^{\infty} a_n^2 + \frac{a_0^2}{2} = \frac{8}{\pi^2} + \frac{4}{\pi^2} \cdot \sum_{n=2}^{\infty} \frac{[(-1)^n+1]^2}{(n^2-1)^2} =$$

$$= \frac{8}{\pi^2} + \frac{16}{\pi^2} \cdot \sum_{k=1}^{\infty} \frac{1}{(4k^2-1)^2}$$

$$\frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \underbrace{\sin^2(x)}_{= \frac{1 - \cos 2x}{2}} \, dx = \frac{1}{\pi} \cdot \pi = 1$$

$$\Rightarrow \quad \sum_{k=1}^{\infty} \frac{1}{(4k^2-1)^2} = \frac{1 - 8/\pi^2}{16/\pi^2} = \frac{\pi^2 - 8}{16} = \frac{\pi^2}{16} - \frac{1}{2}$$

№5

1) False;

e^{2xt-t^2} generates Hermite polynomials in the following way:

$$e^{2xt-t^2} = \sum_{k=0}^{\infty} H_k(x) \frac{t^k}{k!}$$

2) True.

3) False;

consider just 2^{nd} eq.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \implies \dot{y} = y \implies y(t) = C \cdot e^{t}$$

$$\implies |y(t)| \xrightarrow[t \to +\infty]{} \infty$$

(no stability in y-direction

4) True;

$$\underbrace{x(1-x)}_{\Rightarrow\, x=1,\ x=0 \text{ are singular pts.}} y'' + (3x-1)y' + y = 0 \implies y'' + \underbrace{\frac{3x-1}{x(1-x)}}_{=: p(x)} y' + \underbrace{\frac{1}{x(1-x)}}_{=: q(x)} y = 0$$

$$\left. \begin{aligned} &|(x-1) \cdot p(x)| \xrightarrow[x \to 1]{} 2 < \infty \\ &|(x-1)^2 \cdot q(x)| \xrightarrow[x \to 1]{} 0 < \infty \end{aligned} \right\} \implies x = 1 \text{ is a regular singular pt.}$$

5) False;

$$\underbrace{(1-x^2)}_{\Rightarrow\, x = \pm 1 \text{ are singular pts.}} y'' - 2xy' + 6y = 0$$

$x = 0$ is not a singular pt. at all!