Tutorial #1

№ 18.1.8

Need to show: $\sin^2 z + \cos^2 z = 1$

Recall Euler's formulae: $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

Now z is complex: $z = x + iy$

$$\sin z = \frac{e^{ix} \cdot e^{-y} - e^{-ix} \cdot e^{y}}{2i}$$

$$\cos z = \frac{e^{ix} \cdot e^{-y} + e^{-ix} \cdot e^{y}}{2}$$

$$\sin^2 z + \cos^2 z = \frac{1}{4}\left[-\left(e^{ix} \cdot e^{-y} - e^{-ix} \cdot e^{y}\right)^2 + \left(e^{ix} \cdot e^{-y} + e^{-ix} \cdot e^{y}\right)^2\right] =$$

$$= \frac{1}{4}\left[\left(\cancel{e^{ix} \cdot e^{-y}} - e^{ix} \cdot e^{-y} + e^{-ix} \cdot e^{y} + \cancel{e^{-ix} \cdot e^{y}}\right) \cdot \left(\cancel{e^{ix} \cdot e^{-y}} + e^{ix} \cdot e^{-y} + e^{-ix} \cdot e^{y} - \cancel{e^{-ix} \cdot e^{y}}\right)\right] =$$

$$= \frac{\cancel{2} e^{-ix} \cdot e^{y} \cdot \cancel{2} e^{ix} \cdot e^{-y}}{\cancel{4}} = 1$$

№ 18.1.9

$$\ln(1+i) = \left\{ \begin{array}{l} \text{Write } 1+i \text{ in} \\ \text{polar representation} \\ 1+i = \sqrt{2} \cdot e^{i(\pi/4 + 2\pi n)} \\ \quad \underset{\sqrt{x^2+y^2}}{\uparrow} \quad \underset{=\arctan(y/x)}{\uparrow} \\ (x = 1, y = 1 \text{ in this case}) \end{array} \right\} = \ln\sqrt{2} + i\left(\pi/4 + 2\pi n\right) =$$

$$= \frac{1}{2}\ln 2 + \frac{i\pi}{4}(1 + 8n), \quad \underbrace{n \in \mathbb{Z}}_{\text{infinitely many branches}}$$

№ 18.1.20

$$\lim_{z \to \frac{i\pi}{2}} z^4 \cdot \cosh\frac{z}{3} = ?$$

If a limit exists, it doesn't depend on direction along which we evaluate it. In this case, choose a vertical direction, i.e. set $z = 0 + iy$ and let $y \to \pi/2$.

$$\lim_{z \to \frac{i\pi}{2}} z^4 \cdot \cosh\frac{z}{3} = \lim_{y \to \pi/2} (iy)^4 \cdot \cosh\left(\frac{iy}{3}\right) =$$

Recall: $\cosh it = \cos t$, since $\underbrace{\frac{e^{(it)} + e^{-(it)}}{2}}_{= \cosh(it)} = \cos t$

$$= \lim_{y \to \pi/2} y^4 \cdot \cos(y/3) = \frac{\pi^4}{16} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}\,\pi^4}{32}$$