Tutorial # 12

№ 21.5.3

Let $X(t)$ be # of events occuring on the time t, given the mean rate λ.

$$P\left(X(t)=k\right) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Now let our random variable be not # of events, occured on the interval $[0, t]$, but the time T_1 at which the 1^{st} event occurs. Then,

$$P\left(T_1 > t\right) = P\left(X(t)=0\right) = e^{-\lambda t}.$$

$$\Rightarrow P\left(T_1 \leq t\right) = 1 - e^{-\lambda t}$$

$$\Rightarrow p_{T_1}(t) = \frac{d\,P(T_1 \leq t)}{dt} = \lambda e^{-\lambda t}$$

Similarly, let T_m be the time when m^{th} event occurs.

Then, $$P\left(T_m > t\right) = P\left(X(t) < m-1\right) = \sum_{k=0}^{m-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

$$\Rightarrow P\left(T_m \leq t\right) = 1 - \sum_{k=0}^{m-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

$$\Rightarrow p_{T_m}(t) = \frac{d\,P(T_m \leq t)}{dt} = -\frac{d}{dt}\left[\sum_{k=0}^{m-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}\right] = e^{-\lambda t}\left[\sum_{k=0}^{m-1} \frac{\lambda^{k+1} t^k}{k!} e^{-\lambda t} - \underbrace{\sum_{k=1}^{m-1} \frac{\lambda^k t^{k-1}}{(k-1)!}}_{\left(\sum_{k=0}^{m-2} \frac{\lambda^{k+1} t^k}{k!}\right.}\right]$$

$$= e^{-\lambda t} \cdot \frac{\lambda^m \cdot t^{m-1}}{(m-1)!}$$

N 19.1.13

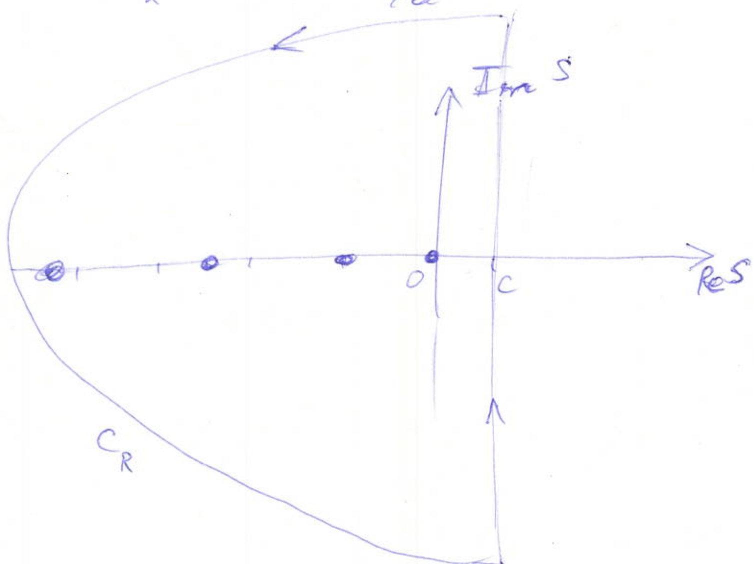
$$\mathcal{L}^{-1}\left[\frac{\cosh(x\sqrt{s})}{s\cosh(a\sqrt{s})}\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \underbrace{\frac{e^{st}\cdot\cosh(x\sqrt{s})}{s\cdot\cosh(a\sqrt{s})}}_{=:f(s)} ds$$

Near $s=0$, $\cosh(\dots)\approx 1$ $\Rightarrow$ it's not a branch pt., but due to the presence of $1/s$ it's still a simple pole.

Other Poles ? : $\cosh(a\sqrt{s}) = 0 \Rightarrow \cos(ia\sqrt{s}) = 0$

$$\Rightarrow ia\sqrt{s} = \frac{\pi}{2} + \pi k = \frac{\pi}{2}\cdot(2k+1), \quad k = 0, \pm1, \pm2, \dots$$

$$\Rightarrow s_k = -\frac{\pi^2(2k+1)^2}{4a^2}, \quad k = 0, 1, 2, \dots$$ poles of 1st order



$$\left|\int_{C_R} \dots ds\right| = \left\{\begin{array}{l} s-c = Re^{i\theta} = \\ = R\cos\theta + iR\sin\theta \\ ds = iRe^{i\theta}d\theta \end{array}\right\} = e^{ct}\int_{\pi/2}^{3\pi/2} \underbrace{e^{+\overbrace{R\cos\theta}^{\leq 0}\cdot t}}_{\to 0 \text{ as } R\to\infty} \cdot \frac{1}{R} \cdot \underbrace{\left|\frac{\cosh(x\sqrt{s})}{\cosh(a\sqrt{s})}\right|}_{\text{may grow at most as } e^{x\sqrt{R}} \text{ that is slowly then the 1st factor}} d\theta \underset{R\to\infty}{\to} 0$$

$$\int_{c-i\infty}^{c+i\infty} \dots + \int_{C_R} \dots = \oint_C \dots = 2\pi i \cdot \sum_{k=0}^{\infty} Res(f; s_k) + Res(f; 0)$$

As $R \to \infty$,

$$\int_{c-i\infty}^{c+i\infty} \ldots = 2\pi i \sum_{k=0}^{\infty} \operatorname{Res}(f; s_k) =$$

$$\operatorname{Res}(f; s_k) = \left.\frac{e^{st} \cdot \cosh(x\sqrt{s})}{[s \cdot \cosh(a\sqrt{s})]'}\right|_{s=s_k} = \frac{e^{s_k t} \cdot \cosh(x\sqrt{s_k})}{\underbrace{\cosh(a\sqrt{s_k})}_{=0} - \frac{a\sqrt{s_k}}{2} \sinh(a\sqrt{s_k})} =$$

$$= -\frac{4}{i\pi(2k+1)} e^{-\frac{\pi^2(2k+1)^2}{4a^2}t} \cdot \frac{\cos\left(\frac{\pi(2k+1)x}{2a}\right)}{-i \underbrace{\sin\left(\frac{\pi(2k+1)}{2}\right)}_{= \cos(\pi k) = (-1)^k}} = -\frac{4(-1)^k}{\pi(2k+1)} e^{-\frac{\pi^2(2k+1)^2}{4a^2}t} \cdot \cos\left(\frac{\pi(2k+1)}{2a}x\right)$$

$$\operatorname{Res}(f; 0) = 1$$

$$= \sum_{k=0}^{\infty} \frac{8 \cdot (-1)^{k+1}}{(2k+1)} \cdot \cos\left(\frac{(2k+1)\pi x}{2a}\right) \cdot e^{-\frac{\pi^2(2k+1)^2}{4a^2}t}$$

$$\mathcal{L}^{-1}\left[\frac{\cosh(x\sqrt{s})}{s\cosh(a\sqrt{s})}\right] = \operatorname{Res}(f;0) + \sum_{k=0}^{\infty} \operatorname{Res}(f; s_k) = 1 - \frac{4}{\pi} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cdot e^{-\frac{\pi^2(2k+1)^2}{4a^2}t} \cdot \cos\left(\frac{(2k+1)\pi x}{2a}\right)$$

$$= 1 + \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1} e^{-\frac{\pi^2(2k-1)^2}{4a^2}t} \cdot \cos\left(\frac{(2k-1)\pi x}{2a}\right) =$$