Tutorial #2

№ 18.2.11

$u(x,y) = x + y$, $v(x,y) = ?$

Recall if $f(z)$ is analytic,
$f(z) = u(x,y) + i\,v(x,y)$, then the Cauchy-Riemann eq. hold:

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v$$

In our case: $\partial_y v = \partial_x u = 1 \Rightarrow v(x,y) = c_1(x) + y$

$$\underbrace{\partial_x v}_{c_1'(x)} = -\partial_y u = -1 \Rightarrow c_1(x) = c_2 - x$$

$\Rightarrow v(x,y) = y - x + c_2$ — arbitrary const.

№ 18.2.17

a) $f(z) = \frac{\sin z^{1/2}}{z^{1/2}}$: $z=0$ is the only candidate,

however, by series expansion about $z=0$: $f(z) = \frac{1}{z^{1/2}} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{\frac{2n+1}{2}} =$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^n$ — all terms have non-negative integer powers of z $\Rightarrow$ $f(z)$ is analytic at $z=0$, even though formally it's undefined $\Rightarrow$ can call $z=0$ a removable singularity.

b) $f(z) = \sec \frac{1}{z} = \frac{1}{\cos(1/z)}$

$\cos(1/z) = 0$ yields simple poles of $f(z)$:

$\frac{1}{z} = \frac{\pi}{2} + \pi n, \quad n \in \mathbb{Z} \quad \Rightarrow z = \frac{2}{\pi(2n+1)}, \quad n \in \mathbb{Z}$

N 18.2.21

Cauchy-Riemann (C.-R.) eq.-s at a point z_0 follow from differentiability of $f(z)$, i.e. existence of $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ which is independent of a way of approaching z_0. Rather than using horizontal/vertical lines, let's use radial/angular directions in evaluation of the limit. This will result in C.-R. eq.-s in polar form.

Let $z = re^{i\theta}$. Assuming $f(z)$ is analytic at $z_0 = r_0 e^{i\theta_0}$, we have:

$$f'(z_0) = \lim_{\substack{r \to r_0 \\ (\theta = \theta_0)}} \frac{f(re^{i\theta_0}) - f(r_0 e^{i\theta_0})}{re^{i\theta_0} - r_0 e^{i\theta_0}} = \lim_{\substack{\theta \to \theta_0 \\ (r = r_0)}} \frac{f(r_0 \cdot e^{i\theta}) - f(r_0 e^{i\theta_0})}{r_0 e^{i\theta} - r_0 \cdot e^{i\theta_0}}$$

Since $f(z) = u(r, \theta) + i v(r, \theta)$, we evaluate:

$$f'(z_0) = \lim_{r \to r_0} \frac{u(r, \theta_0) + i v(r, \theta_0) - u(r_0, \theta_0) - i v(r_0, \theta_0)}{(r - r_0) \cdot e^{i\theta_0}} =$$

$$= e^{-i\theta_0} \left[\lim_{r \to r_0} \frac{u(r, \theta_0) - u(r_0, \theta_0)}{r - r_0} + i \lim_{r \to r_0} \frac{v(r, \theta_0) - v(r_0, \theta_0)}{r - r_0} \right] =$$

$$= e^{-i\theta_0} \cdot \left[\partial_r u(r_0, \theta_0) + i\, \partial_r v(r_0, \theta_0) \right]$$

On the other hand:

$$f'(z_0) = \lim_{\theta \to \theta_0} \frac{u(r_0,\theta) + i v(r_0,\theta) - u(r_0,\theta_0) - i v(r_0,\theta_0)}{r_0 \cdot \underbrace{(e^{i\theta} - e^{i\theta_0})}_{\approx\, i e^{i\theta_0}(\theta-\theta_0)}} =$$

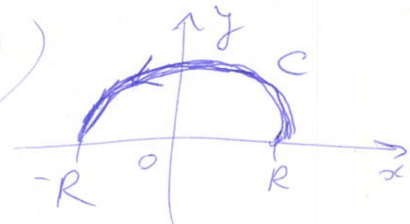
$$= \frac{e^{-i\theta_0}}{i r_0} \cdot \left[\lim_{\theta \to \theta_0} \frac{u(r_0,\theta) - u(r_0,\theta_0)}{\theta - \theta_0} + i \lim_{\theta \to \theta_0} \frac{v(r_0,\theta) - v(r_0,\theta_0)}{\theta - \theta_0} \right] =$$

$$= \frac{e^{-i\theta_0}}{r_0} \cdot \left[-i \cdot \partial_\theta u(r_0,\theta_0) + \partial_\theta v(r_0,\theta_0) \right]$$

Therefore,
$$\begin{cases} \partial_r u(r_0,\theta_0) = \dfrac{1}{r_0} \partial_\theta v(r_0,\theta_0) \\ \partial_\theta u(r_0,\theta_0) = -r_0 \partial_r v(r_0,\theta_0) \end{cases}$$

N 18.3.12

$$\left| \int_C \frac{dz}{1+z^2} \right| \le \int_C \frac{|dz|}{|1+z^2|} = \begin{pmatrix} z = R \cdot e^{i\theta} \\ dz = iRe^{i\theta} \end{pmatrix}$$



$$= \int_0^\pi \frac{R \, d\theta}{\underbrace{|1 + R^2 \cos 2\theta + iR^2 \sin 2\theta|}} \le \frac{\pi R}{|R^2 - 1|} \xrightarrow[R \to \infty]{} 0$$

$$= \sqrt{(1+R^2\cos 2\theta)^2 + R^4 \sin^2 2\theta} = \sqrt{1 + R^4 + 2R^2 \cos 2\theta} \ge |R^2 - 1|$$ when $\cos 2\theta = -1$

$$\left\{ z^2 = R^2 e^{2i\theta} = R^2 \cos 2\theta + iR^2 \sin 2\theta \right\}$$

N 18. 3. 15

$$\oint_C \frac{dz}{(z-a)^n} = \left(\begin{array}{l} z-a = R\cdot e^{i\theta} \\ dz = iRe^{i\theta} \end{array} \right) =$$

if $n \neq 1$

$$= \int_0^{2\pi} \frac{i e^{i\theta(1-n)}}{R^{n-1}} \, d\theta = -\frac{1}{(n-1)\cdot R^{n-1}} \, e^{-i\theta(n-1)} \Big|_0^{2\pi} =$$

$$= 0 \quad \text{for} \quad n = 2, 3, \ldots$$

If $n = 1$, then:

$$\int_0^{2\pi} i \, d\theta = 2\pi i$$