Tutorial #3

№ 18.4.1

Let's compare differentiability of functions of real and complex arguments. Consider $f(x) = x^3 \sin(1/x)$ as function of real variable x. It's differentiable at $x=0$, but only once:

$$f'(x) = 3x^2 \cdot \sin(1/x) - x\cdot\cos\left(\frac{1}{x}\right) \Rightarrow f'(0)=0$$

$$f''(x) = 6x\cdot\sin(1/x) - 4\cos(1/x) - \underbrace{\frac{1}{x}\cdot\sin(1/x)}_{\substack{\to\infty \\ \text{as } x\to 0}}$$

This is in contrast with functions of complex variable which are differentiable infinite number of times if differentiable at all. Even though every real function $F(x)$ can be thought as restriction of a function of complex variable $F(z)$ to the real line $\left(F(x) = F(z)\Big|_{y=0} = F(x+iy)\Big|_{y=0}\right)$, there's no paradox since differentiability condition is much more restrictive. In our case $f(z) = z^3\cdot\sin(1/z)$ would not be differentiable at all in complex plane.

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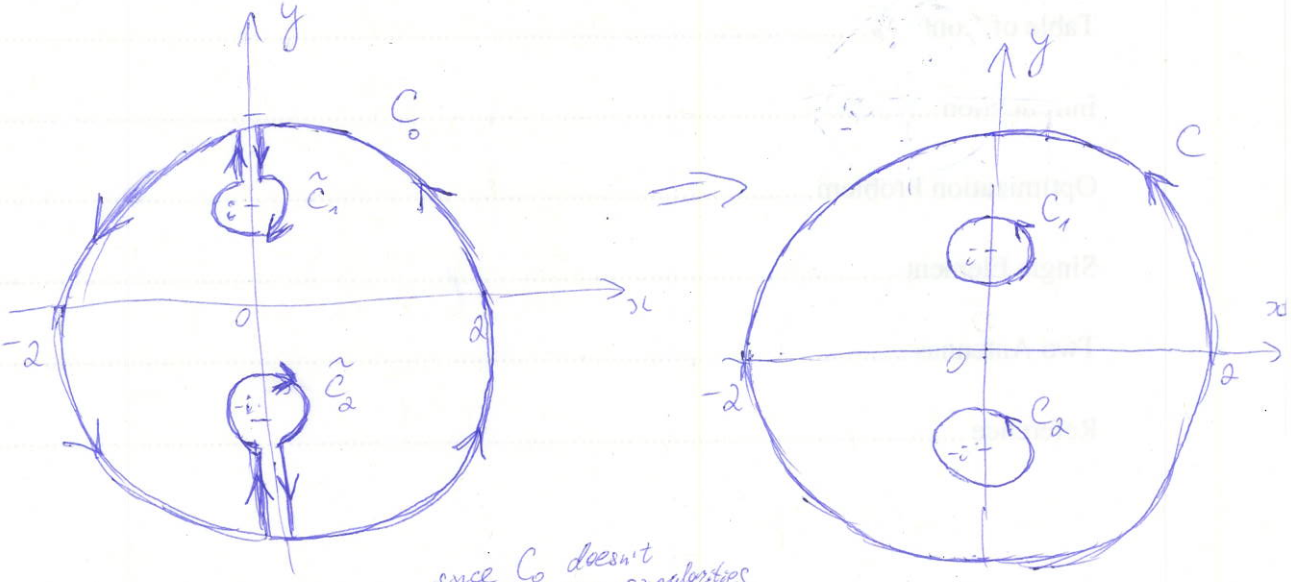
$$I = \oint\limits_{C:|z|=2} \frac{e^{iz}}{z^2+1}\,dz = \left\{ \begin{aligned} &\frac{1}{1+z^2} = \frac{1}{(1-iz)(1+iz)} = \\ &= \frac{1/2}{1-iz} + \frac{1/2}{1+iz} = \\ &= \frac{i}{2}\left(\frac{1}{z+i} - \frac{1}{z-i}\right) \end{aligned} \right\} = \frac{i}{2}\oint\limits_C \overbrace{\frac{e^{iz}}{z+i}\,dz}^{\substack{\text{by Cauchy's integral formula} \\ \left(= 2\pi i\cdot e^{iz}\big|_{z=-i}\right.}} - \frac{i}{2}\oint\limits_C \overbrace{\frac{e^{iz}}{z-i}\,dz}^{= 2\pi i\cdot e^{iz}\big|_{z=i}} =$$

$$= -\pi e + \pi e^{-1} = \frac{\pi(1-e^2)}{e}$$

Alternatively, instead of partial fraction expansion and direct application of the Cauchy's integral formula, we will deform original contour C (by the Cauchy's theorem).



since C_0 doesn't contain any singularities

$$\oint_{C_0} \ldots = 0 \quad \Longrightarrow \quad \oint_{C} \ldots = \oint_{C_1} \ldots + \oint_{C_2} \ldots$$

$$\Leftarrow \oint_{C_0} \ldots + \oint_{\tilde{C}_1} \ldots + \oint_{\tilde{C}_2} \ldots = 0$$

since $\oint_{C_1} \ldots = -\oint_{\tilde{C}_1}$, $\oint_{C_2} \ldots = -\oint_{\tilde{C}_2} \ldots$

$$I = \oint_{C_1} \frac{e^{iz}}{z^2+1}\,dz + \oint_{C_2} \frac{e^{iz}}{z^2+1}\,dz = \left\{ z^2+1 = (z-i)(z+i) \right\}$$

$$= \oint_{C_1} \frac{e^{iz}/(z+i)}{z-i}\,dz + \oint_{C_2} \frac{e^{iz}/(z-i)}{z+i}\,dz = 2\pi i \cdot \left.\frac{e^{iz}}{z+i}\right|_{z=i} +$$

$$+ 2\pi i \left.\frac{e^{iz}}{z-i}\right|_{z=-i} = \pi e^{-1} - \pi e = \frac{\pi(1-e^2)}{e}$$

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Introduction

Federal Communication Commission has specified the frequency range of 3.1-10.6 GHz to the UWB systems [1] and UWB technology has been applied in many systems since it can provide capability of high data rate in short range communications [2-3]. Although according to the FCC regulations the UWB systems are limited in power to -41.3 dBm/MHz [1], but using Multi Input Multi Output (MIMO) technique in UWB systems can make the best possible use of radiated and received power. It has been reported that in a rich scattering environment, capacity of a communication system with M receivers and N transmitter antennas is more than a Single Input Single Output (SISO) system by a factor of min(M,N) [3-4]. Also Using MIMO technique is more efficient for the transmitted power [3].

Using antenna elements in small handhelds to apply MIMO techniques will affect the performance of the antennas. Also to be able to use the MIMO technique, the antennas needs to be decoupled from each other, which is the next problem in for using the antennas in small handheld devices. There are many techniques for increasing isolation between the antenna elements. Adding parasitic elements like a tree-like structure between the driven antennas has been used to increase isolation between antennas in Figure 1 [5].

$$I = \oint \frac{\sin z}{z^{n+1}}\,dz = \frac{2\pi i}{n!}\cdot\frac{d^n}{dz^n}(\sin z)\Big|_{z=0} =$$

$C: |z|=2$

(where n is positive integer)

$$= \begin{cases} \frac{2\pi i}{n!}\cdot(-1)^{\frac{n-1}{2}}\cdot \overbrace{\cos \overset{0}{z}}^{=1}\Big|_{z=0} = \frac{2\pi i}{n!}(-1)^{\frac{n-1}{2}}, & n - \text{odd} \\ \frac{2\pi i}{n!}\cdot 0 = 0, & n - \text{even} \end{cases}$$

we have used a more general version of the Cauchy integral formula:

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz, \quad n=0,1,2,\ldots$$

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This is an application to PDE.

Note: If $f(x+iy) = u(x,y) + iv(x,y)$ is analytic,

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases} \Bigg| \begin{matrix} \partial_x \\ + \\ \partial_y \end{matrix} \Rightarrow \Delta u = \partial^2_{xy} v - \partial^2_{yx} v = 0$$

Cauchy-Riemann eq.-s

(similarly, $\Delta v = 0$)

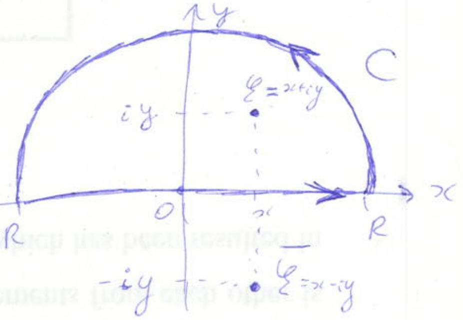
Assume we want to solve the Laplace eq. for upper half-plane:

$$\begin{cases} \Delta u(x,y) = 0 \quad , \quad x \in \mathbb{R}, \ y > 0. \\ u(x,0) = \underbrace{g(x)}_{\text{given function}} \end{cases}$$

We expect solution to be:

$$u(x,y) = \int_{-\infty}^{+\infty} K(\tilde{x},y) \cdot g(\tilde{x}) \, d\tilde{x}$$

Try to obtain this from Cauchy integral formula for f and contour C:



$$\ominus \begin{cases} f(\xi) = \frac{1}{2\pi i} \oint_C \frac{f(\tilde{z})}{\tilde{z} - \xi} \, d\tilde{z} \\ 0 = \frac{1}{2\pi i} \cdot \oint_C \frac{f(\tilde{z})}{\tilde{z} - \xi} \, d\tilde{z} \quad \left(\text{since } \frac{f(\tilde{z})}{\tilde{z}-\bar{\xi}} \text{ is analytic in } C\right) \end{cases}$$

$$\Rightarrow f(\xi) = \frac{1}{2\pi i} \oint_C \frac{f(\tilde{z}) \cdot (\xi - \bar{\xi})}{(\tilde{z}-\xi) \cdot (\tilde{z} - \bar{\xi})} \, d\tilde{z} = \frac{1}{2\pi i} \cdot \int_{-R}^{R} \frac{f(\tilde{x}) \cdot (\xi - \bar{\xi})}{(\tilde{x}-\xi)(\tilde{x}-\bar{\xi})} \, d\tilde{x} \ +$$

$$+ \underbrace{\frac{1}{2\pi i} \cdot \int_{\underset{\text{half-circumference}}{C_R}} \frac{f(\tilde{z}) \cdot (\xi - \bar{\xi})}{(\tilde{z}-\xi) \cdot (\tilde{z}-\bar{\xi})} \, d\tilde{z}}_{} \ \xrightarrow[R\to\infty]{} \ \frac{1}{2\pi i} \cdot \int_{-\infty}^{+\infty} \frac{f(\tilde{x}) \cdot (\xi - \bar{\xi})}{(\tilde{x}-\xi)(\tilde{x}-\bar{\xi})} \, d\tilde{x}$$

$\xrightarrow[R\to\infty]{} 0$:

$$\left| \int_{C_R} \frac{f(\tilde{z})(\xi - \bar{\xi})}{(\tilde{z}-\xi) \cdot (\tilde{z}-\bar{\xi})} \, d\tilde{z} \right| \le \overbrace{|\xi - \bar{\xi}|}^{=2y} \cdot \int_0^{\pi} \frac{\overbrace{|f(Re^{i\theta})|}^{\le M} \cdot R \, d\theta}{\left| R^2 \underbrace{\left(1 - \frac{\xi e^{-i\theta}}{R}\right)\left(1 - \frac{\bar{\xi} e^{-i\theta}}{R}\right)}_{\to 1 \text{ as } R\to\infty} \right|}$$

$$\le \frac{2yM}{R} \cdot \pi \longrightarrow 0 \text{ as } R \to \infty$$

Hence,

$$f(x,y)=f(\xi)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{f(\tilde{x},0)\overbrace{(\xi-\bar{\xi})}^{=2iy}}{\underbrace{(\tilde{x}-\xi)(\tilde{x}-\bar{\xi})}_{=(\tilde{x}-x-iy)(\tilde{x}-x+iy)=(\tilde{x}-x)^2+y^2}}\,d\tilde{x}=$$

$$=\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{f(\tilde{x},0)}{(\tilde{x}-x)^2+y^2}\,d\tilde{x}$$

If we write $f(x,y)=u(x,y)+iv(x,y)$, then:

$$u(x,y)=\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{u(\tilde{x},0)}{(\tilde{x}-x)^2+y^2}\,d\tilde{x}=\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{g(\tilde{x})\,d\tilde{x}}{(\tilde{x}-x)^2+y^2}$$

This is the Poisson integral formula for half-plane.

Also there is the idea of using two quarter wavelength slits on the ground plane to make a transmission zero in the coupling path between the antenna elements in Figure 3. is interesting but it`s not general [10].

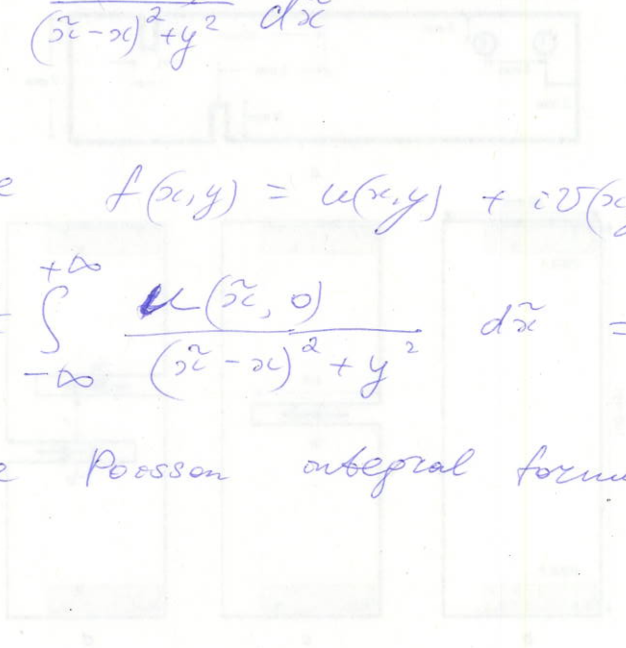


Figure 3: Layout

(a) Single PIFA element

(b) Two-element array on handheld (no slits on the ground plane)

(c) Coupling reduction scheme by inserting single slit (notch) on ground plane

(d) Coupling reduction scheme by inserting two coupled slits at distance d from each other [10]

Printed Circuit Monopoles (PCM) are good candidates for implementation in of UWB handheld devices [11]. They have shown good impedance bandwidth, high radiation efficiency, linear phase response, and stable radiation pattern in the working band [12]. In this paper using microstrip monopole antenna and defining an optimization problem we design MIMO antennas for UWB communications. During the optimization process we minimize the spacing between

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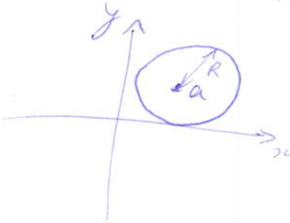
Let $f(z)$ be analytic in $\mathcal{D}: |z-a| \le R$ (disk of rad. R centered at $z=a$)

Then if $|f(z)| \le M$, we have:



$$|f^{(n)}(a)| \le \frac{n!M}{R^n}, \quad n = 0, 1, 2, \ldots$$

Cauchy's inequality

Indeed, By the Cauchy integral formula:

$$|f^{(n)}(a)| = \left|\frac{n!}{2\pi i}\cdot\oint_{C:|z-a|=R}\frac{f(z)\,dz}{(z-a)^{n+1}}\right| \le \frac{n!}{2\pi}\int_0^{2\pi}\frac{\overbrace{|f(a+Re^{i\theta})|}^{\le M}\cdot\overbrace{|Rie^{i\theta}|}^{=R}d\theta}{R^{n+1}} =$$

$$\left\{\begin{array}{l} z-a = Re^{i\theta} \\ dz = iRe^{i\theta}d\theta \end{array}\right\}$$

$$= \frac{n!}{2\pi}\cdot\frac{M}{R^n}\int_0^{2\pi}d\theta = \frac{n!\cdot M}{R^n}, \quad n=0,1,2,\ldots$$

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Liouville's th.: if an entire (i.e. analytic in the whole complex plane) function $f(z)$ is bounded, then it must be a constant.

Let a be arbitrary pt. in the complex plane. Then, by analyticity,

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(z-a)^n \Rightarrow |f(z)| \le |f(a)| + \sum_{n=1}^{\infty}\frac{|f^{(n)}(a)|}{n!}\cdot|z-a|^n$$

By the result of the prev. problem: $\frac{|f^{(n)}(a)|}{n!} \le \frac{M}{R^n} \xrightarrow[\text{as } R\to\infty]{} 0$ (since we can choose arbitrary big disk)

$$\Rightarrow |f(z)| \le |f(a)| \Rightarrow f(z) \equiv \text{const}$$