Tutorial # 4

№ 19.2.4

$$\int_0^{\pi} \frac{d\theta}{\cos^2\theta + 4\sin^2\theta} = \left\{ \begin{array}{l} 2\theta =: \varphi \\ d\theta = d\varphi/2 \end{array} \right\} = \int_0^{2\pi} \frac{d\varphi}{5 - 3\cos\varphi} =$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \left\{ \begin{array}{l} z := e^{i\varphi} \\ dz = i e^{i\varphi} d\varphi \\ \cos\varphi = \frac{z + 1/z}{2} = \frac{z^2+1}{2z} \end{array} \right\} = \oint_{C:|z|=1} \frac{2/i \; dz}{10z - 3(z^2+1)} \;(=)$$

Where are the poles?

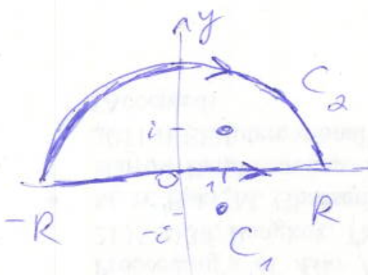
$$3z^2 - 10z + 3 = 0 \Rightarrow z_{1,2} = \frac{5}{3} \pm \frac{\overbrace{\sqrt{25-9}}^{=\frac{4}{3}}}{3}$$

Only $z = 1/3$ pole is inside C.

$$(=)\; \frac{4\pi}{3} \cdot \frac{\cancel{\left(z - \frac{1}{3}\right)}}{\left(z - \frac{1}{3}\right)\left(z - 3\right)} \Bigg|_{z=\frac{1}{3}} = \frac{4\pi}{3 \cdot 8/3} = \frac{\pi}{2}$$

N 19.2.15

$$\int_{-\infty}^{\infty} \frac{\cos \pi x \, dx}{x^2-2x+2} = \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{i\pi x}}{x^2-2x+2}\,dx =$$



$$z^2-2z+2=0$$

$$z_{1,2} = 1 \mp i$$

in the limit as $R \to +\infty$

$$= \mathrm{Re}\int_{C_1}\dots = \mathrm{Re}\left[-\int_{C_2} + 2\pi i \cdot \mathrm{res}(1+i)\right] =$$

$$\left|\int_{C_2} \frac{e^{i\pi z}}{z^2-2z+2}\,dz\right| = \left\{ \begin{array}{l} z = R\cdot e^{i\theta} \\ dz = iR\cdot e^{i\theta}\cdot d\theta \end{array} \right\} = \left|\int_{\pi}^{0} \frac{R\cdot e^{i\pi R(\cos\theta + i\sin\theta)}}{R^2\cdot e^{2i\theta} - 2Re^{i\theta}+2}\, d\theta\right|$$

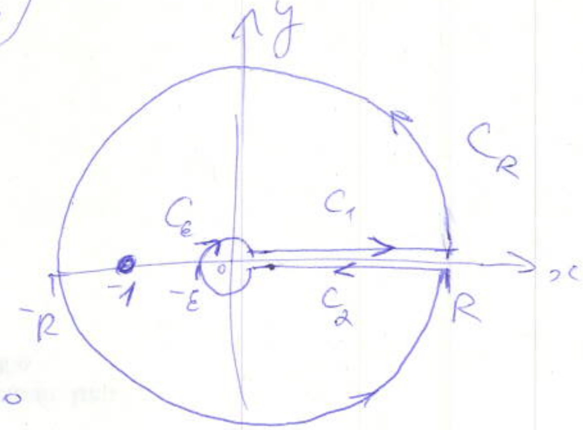
$$\le \int_0^\pi \frac{R\cdot e^{\overbrace{-\pi R\sin\theta}^{\to 0}}}{\underbrace{|R^2 e^{2i\theta} - 2Re^{i\theta}+2|}}\, d\theta \to 0 \quad \text{as } R \to +\infty$$

$$= \mathrm{Re}\left[2\pi i \cdot \mathrm{res}\,(1+i)\right] = \mathrm{Re}\left[2\pi i \cdot \frac{\cancel{(z-(1+i))}\, e^{i\pi z}}{(z-(1-i))\cancel{(z-(1+i))}}\Bigg|_{z=1+i}\right] =$$

$$= \mathrm{Re}\left[\pi \cdot e^{i\pi}\cdot e^{-\pi}\right] = -\pi e^{-\pi}$$

N 19.2.23

$$I = \int_0^{\infty} \frac{x^{1/2}}{(1+x)^2}\, dx =$$



in the limit as $R \to +\infty$, $\varepsilon \to 0$

$$= \int_{C_1} \ldots = 2\pi i \operatorname{res}(z=-1) - \int_{C_R} \ldots - \int_{C_\varepsilon} \ldots - \int_{C_2} \ldots$$

$$\left.\frac{d\, z^{1/2}}{dz}\right|_{z=-1} = -\frac{i}{2}$$

$$\left| \int_{C_R} \frac{z^{1/2}}{(1+z)^2}\, dz \right| = \left\{ \begin{aligned} z &= Re^{i\theta} \\ dz &= iRe^{i\theta} d\theta \end{aligned} \right\} = \left| \int_0^{2\pi} \frac{\sqrt{R} \cdot R e^{i\theta/2}}{(1+Re^{i\theta})^2}\, d\theta \right| \leq$$

$$\leq \int_0^{2\pi} \underbrace{\frac{R^{3/2}}{\left|(1+Re^{i\theta})^2\right|}}_{\sim \frac{1}{R^{1/2}} \to 0 \text{ as } R\to\infty} d\theta$$

$$\left| \int_{C_\varepsilon} \frac{z^{1/2}}{(1+z)^2}\, dz \right| \leq \left\{ \begin{aligned} z &= \varepsilon \cdot e^{i\theta} \\ dz &= i\varepsilon e^{i\theta} \end{aligned} \right\} \leq \int_{-2\pi}^{0} \underbrace{\frac{\varepsilon^{3/2}}{(1+\varepsilon e^{i\theta})^2}}_{\sim \varepsilon^{3/2} \to 0 \text{ as } \varepsilon \to 0} d\theta$$

$$\int_{C_2} \frac{z^{1/2}}{(1+z)^2}\, dz = \left\{ \begin{aligned} z &= x \cdot e^{2\pi i} \\ dz &= dx \cdot e^{2\pi i} \\ z^{1/2} &= \sqrt{x} \cdot e^{\pi i} = -\sqrt{x} \end{aligned} \right\} \overset{\text{as } R\to\infty}{=} -\int_{+\infty}^{0} \frac{\sqrt{x}\, dx}{(1+x)^2} = \int_0^{+\infty} \frac{\sqrt{x}\, dx}{(1+x)^2} = I$$

$$I = \pi - 0 - 0 - I \Rightarrow I = \frac{\pi}{2}$$

№ 19.3.3

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \, ? \qquad a \neq 0, \pm 1, \pm 2, \dots$$

Consider $$\oint_{\square} \pi \cdot \frac{1}{(z+a)^2} \cdot \underbrace{\cot \pi z}_{\text{has simple poles at } z = 0, \pm 1, \pm 2, \dots} \, dz =$$

$\square$ – big square

$$= 2\pi i \left[\sum_{n=-\infty}^{+\infty} \underbrace{\mathrm{res}\,(z=n)}_{= \lim_{z \to n} (z-n)\pi \cdot \frac{\cot \pi z}{(z+a)^2} = \frac{1}{(n+a)^2}} + \overbrace{\mathrm{res}\,(z=-a)}^{= \pi \cdot \frac{d \cot \pi z}{dz}\Big|_{z=-a} = -\frac{\pi^2}{\sin^2 \pi a}} \right] =$$

$$= 2\pi i \left(\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} - \frac{\pi^2}{\sin^2 \pi a} \right) = 0$$

(since as size of the square grows, $\frac{1}{(z+a)^2}$ decays $\Rightarrow \oint_{\square} \dots \to 0$)

$$\Rightarrow \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2 \pi a}$$