Tutorial #5

№ 19.4.5

number of roots (zeros of $f(z)$)

$\underbrace{z^3+z+1}_{=f(z)} = 0 : Z - ?$ for $|z| \leq 1$ or 2.

two separate cases

$f'(z) = 3z^2 + 1$ ← not important

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_C \frac{3z^2+1}{z^3+z+1}\,dz = Z - \underset{\text{number of poles within } C}{\overset{=0}{P}} =$$

$$= \boxed{\frac{1}{2\pi}\Delta_C \arg f(z) = Z}$$

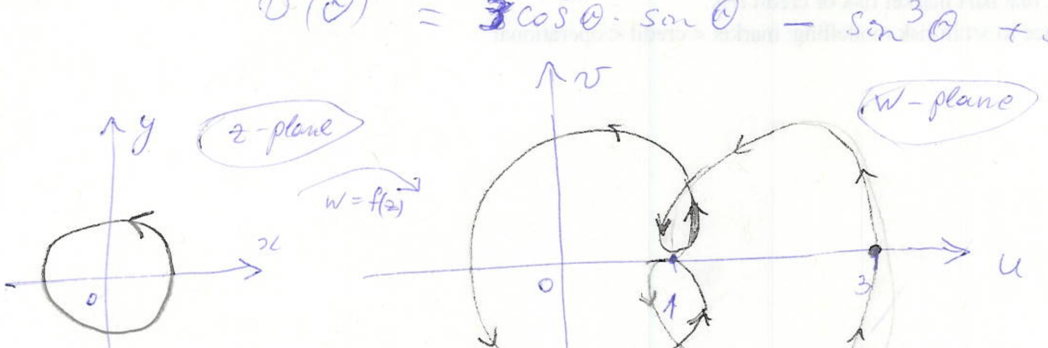
– argument principle (when there're no poles)

$f(z) = f(x+iy) = u(x,y) + iv(x,y)$

1) $|z| \leq 1$: On C: $z = \overset{x}{\cos\theta} + i\overset{y}{\sin\theta}$, $\theta \in [0, 2\pi]$.

$u(\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta + \cos\theta + 1$

$v(\theta) = 3\cos^2\theta\cdot\sin\theta - \sin^3\theta + \sin\theta$



$\Delta_C \arg f(z) = 2\pi$

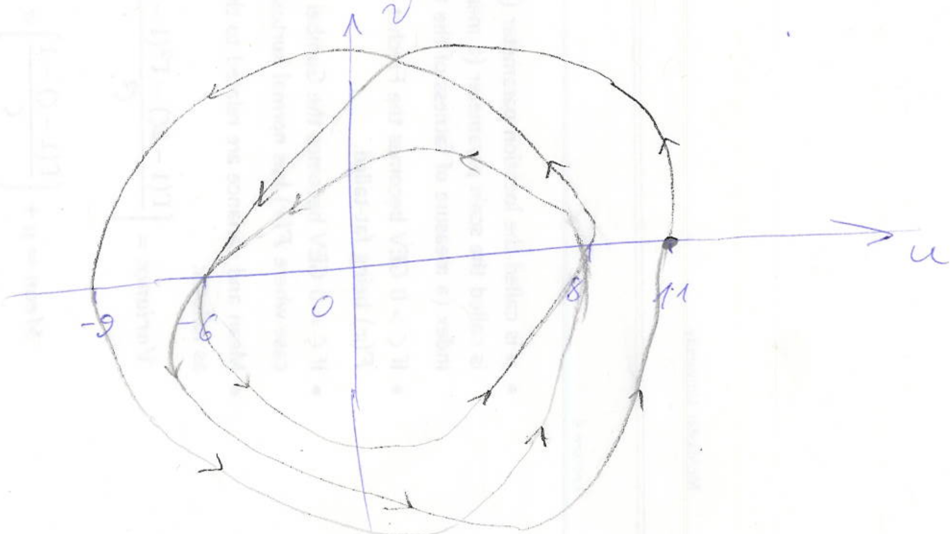
$Z = 1$,

i.e. there's only 1 root inside the unit disk $|z| < 1$

2) $|z| \le 2$: On C: $z = 2\cdot\cos\theta + i\cdot 2\sin\theta$, $\theta \in [0, 2\pi]$

$$u(\theta) = 8\cos^3\theta - 24\cos\theta\cdot\sin^2\theta + 2\cos\theta + 1$$

$$v(\theta) = 24\cos^2\theta\sin\theta - 8\sin^3\theta + 2\sin\theta$$



$\Delta_C \arg f(z) = 6\pi$

$Z = 3$

N 19.4.12

We verify that the equation

$z^7 - 4z^3 + z + 1 = 0$ has 3 zeros in $C: |z| = 1$.

Rouche's th.: Let $f(z), g(z)$ be analytic inside and on a closed curve C, and $|f(z)| > |g(z)|$ on C. Then $f(z)$, $f(z) + g(z)$ have the same number of zeros within C.

Let $f(z) = -4z^3$, $g(z) = z^7 + z + 1$. Then:

$$|g(z)| \le |z|^7 + |z| + 1 = 3 < |f(z)| = 4$$

by triangle ineq. since on $C: |z| = 1$

So Rouche's th. applies:

$f(z) + g(z) = z^7 - 4z^3 + z + 1 = 0$ has the same number of zeros as $f(z) = -4z^3 = 0$ within $C: |z| = 1$

But, clearly, $f(z)$ has a zero $z = 0$ of multiplicity 3, so $z^7 - 4z^3 + z + 1 = 0$ has exactly 3 roots within $C: |z| = 1$.

N 19.4.9

We intend to use argument principle to prove Rouche's th. Therefore, we would like to relate

$\frac{f'(z)+g'(z)}{f(z)+g(z)}$ to $\frac{f'(z)}{f(z)}$, where f, g are analytic on/in C and $|f| > |g|$ on C.

Consider

$$\frac{f'+g'}{f+g} = \frac{f'+g'}{f\cdot\left(1+\frac{g}{f}\right)} = \frac{f'}{f} + \frac{\overbrace{h}^{\text{want to have } \sim\ '?}}{1+g/f} =$$

$$= \frac{f' + f'/f \cdot g + hf}{f\cdot(1+g/f)} \Rightarrow g' = \frac{f'}{f}g + hf$$

$$\Rightarrow h = \frac{g'f - f'g}{f^2} = \left(\frac{g}{f}\right)' = \left(1+\frac{g}{f}\right)'$$

$$\Rightarrow \frac{f'+g'}{f+g} = \frac{f'}{f} + \frac{\left(1+g/f\right)'}{1+g/f}$$

By the argument principle, since f, $f+g$ are analytic ($\Rightarrow$ no poles), we have:

$$\underset{\text{number of zeros of } f+g}{Z_{f+g}} = \underset{\text{number of zeros of } f}{Z_f} + \underbrace{\frac{1}{2\pi i}\oint_C \frac{\left(1+g/f\right)'}{1+g/f}\,dz}_{=\frac{1}{2\pi}\Delta_C\left(1+g(z)/f(z)\right) = 0} = Z_f$$

since $\left|\left(1+g/f\right)-1\right| = |g/f| = \frac{|g|}{|f|} < 1$ and hence contour C is mapped ~~by~~ $1+g(z)/f(z)$ into a contour not encircling the origin (since $1+g/f$ is always inside unit circle centered at $z=1$)