$\sim$ 19.7.5-6

Tutorial # 7

Complex potential:

$$\Omega(z) = \Omega(x,y) = \underset{\text{velocity potential}}{\varphi(x,y)} + i\underset{\text{stream function}}{\psi(x,y)} \quad \text{– analytic function}$$

Velocity field $\vec{v} = \nabla\varphi$ is normal to the curves $\varphi(x,y) = c_1$ (const)

$\Rightarrow$ tangent to the curves ("streamlines") $\psi(x,y) = c_2$

by analyticity of Ω, since

$$\begin{cases} \dfrac{\partial\varphi}{\partial x} + \dfrac{\partial\varphi}{\partial y}\cdot\dfrac{dy}{dx} = 0 \Rightarrow \left(\dfrac{dy}{dx}\right)_\varphi = -\dfrac{\partial\varphi/\partial x}{\partial\varphi/\partial y} \\ \dfrac{\partial\psi}{\partial x} + \dfrac{\partial\psi}{\partial y}\cdot\dfrac{dy}{dx} \Rightarrow \left(\dfrac{dy}{dx}\right)_\psi = -\dfrac{\partial\psi/\partial x}{\partial\psi/\partial y} \end{cases}$$

using C.-R. eq.

$$\left(\frac{dy}{dx}\right)_\varphi\cdot\left(\frac{dy}{dx}\right)_\psi = \frac{\overset{=\frac{\partial\psi}{\partial y}}{\partial\varphi/\partial x}\cdot\overset{=-\frac{\partial\varphi}{\partial y}}{\partial\psi/\partial x}}{\dfrac{\partial\varphi}{\partial y}\cdot\dfrac{\partial\psi}{\partial y}} = -1$$

Let's show $\underset{\text{speed of fluid element}}{v} = \left|\dfrac{d\Omega}{dz}\right|$.

First, we demonstrate that $\dfrac{d\Omega}{dz} = v_x - i v_y$ (components of velocity).

Consider $d\Omega = \dfrac{\partial\varphi}{\partial x}dx + \dfrac{\partial\varphi}{\partial y}dy + i\underbrace{\dfrac{\partial\psi}{\partial x}}_{=-\frac{\partial\varphi}{\partial y}}dx + i\underbrace{\dfrac{\partial\psi}{\partial y}}_{=\frac{\partial\varphi}{\partial x}}dy$

$$= \frac{\partial\varphi}{\partial x}(dx + i dy) + \frac{\partial\varphi}{\partial y}\underbrace{(dy - i dx)}_{=-i\cdot(dx+idy)} = \Big(\underbrace{\frac{\partial\varphi}{\partial x}}_{=v_x} - i\underbrace{\frac{\partial\varphi}{\partial y}}_{=v_y}\Big)dz$$

Now, $\left|\dfrac{d\Omega}{dz}\right| = |v_x - i v_y| = \sqrt{v_x^2 + v_y^2} = v$

№ 19.7.7

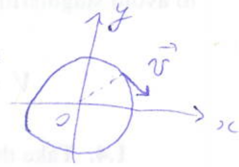
$\Omega(z) = ik \ln z$

Let $z = re^{i\theta} \Rightarrow \ln z = \ln r + i(\theta + 2\pi n)$, $n = 0, \pm 1, \pm 2, \dots$

$\Omega(z) = i\underbrace{k \ln r}_{=\psi(z)} \underbrace{- k(\theta + 2\pi n)}_{=\varphi(z)}$

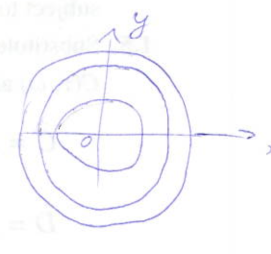
$v_r = \dfrac{\partial \varphi}{\partial r} = 0$, $v_\theta = \dfrac{1}{r}\dfrac{\partial \varphi}{\partial \theta} = -k < 0$ if $k > 0$, clockwise motion

$\Rightarrow$ the motion is circular (vortex motion):



The same is seen from streamlines:

$\psi = k \ln r = \text{const}$ when $r = \text{const}$.



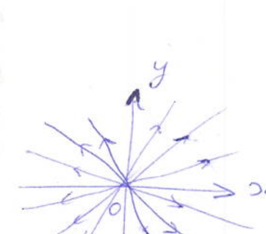
№ 19.7.8

$\Omega(z) = k \ln z$

Again, $z = re^{i\theta} \Rightarrow \ln z = \ln r + i(\theta + 2\pi n)$, $n = 0, \pm 1, \pm 2$

$\Omega(z) = \underbrace{k \ln r}_{=\varphi(z)} + i\underbrace{k(\theta + 2\pi n)}_{=\psi(z)}$

$v_\theta = \dfrac{1}{r}\dfrac{\partial \varphi}{\partial \theta} = 0$, $v_r = \dfrac{\partial \varphi}{\partial r} = \dfrac{k}{r} > 0$ if $k > 0$



and

$\psi(z) = k(\theta + 2\pi n) = \text{const}$ when $\theta = \text{const}$

$\Rightarrow$ streamlines are rays emanating from the origin.