Tutorial # 8

№ 21.1.7

Let X, Y be independent random variables.

$$E\Big[\big(X-\underbrace{\mu_x}_{const}\big)\big(Y-\underbrace{\mu_y}_{const}\big)\Big] = \underbrace{E[X\cdot Y]}_{= E[X]\cdot E[Y] \text{ - by independence}} - \mu_y \overbrace{E[X]}^{=\mu_x} - \mu_x \overbrace{E[Y]}^{=\mu_y} + \mu_x \cdot \mu_y =$$

$$= \underbrace{E[X]}_{=\mu_x} \cdot \underbrace{E[Y]}_{=\mu_y} - \mu_x \mu_y = 0$$

№ 21.1.8

Again, let X, Y be independent.

$$\underbrace{Var[X+Y]}_{= E[(X+Y - \mu_{X+Y})^2]} = \underbrace{E\big[\overbrace{(X+Y)^2}^{=X^2+2XY+Y^2}\big]}_{=E[X^2]+2E[X]\cdot E[Y]+E[Y^2]} - \big(\overbrace{E[X+Y]}^{=E[X]+E[Y]}\big)^2 =$$

$$= E[X^2] - \big(E[X]\big)^2 + E[Y^2] - \big(E[Y]\big)^2 + \cancel{2E[X]\cdot E[Y] - 2E[X]\cdot E[Y]}$$

$$= Var[X] + Var[Y]$$

N 21.1.12

1D random walk:

Place a particle at the origin. Toss a coin and move a particle 1 step to the right if a toss yields heads and 1 to the left if it's tails.

$P(n, m) = ?$ – probability to find the particle m steps from the origin after n tosses (m may be positive or negative)

Let n_h, n_t be numbers of heads and tails, respectively, out of n tosses.

Then: $\begin{cases} n_h - n_t = m \\ n_h + n_t = n \end{cases} \Rightarrow n_h = \frac{n+m}{2}, \quad n_t = \frac{n-m}{2}$ $\Rightarrow$ if n is odd, m has to be odd as well; if n is even, m is even too.

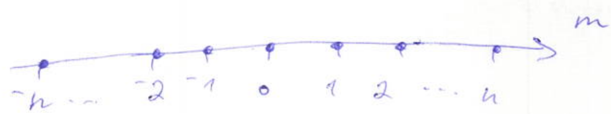
According to binomial distribution:

$$P(n_h) = P(n_t) = \frac{n!}{n_h!(n-n_h)!}\left(\frac{1}{2}\right)^{n_h}\cdot\underbrace{\left(1-\frac{1}{2}\right)^{n-n_h}}_{=\frac{1}{2}} = \frac{n!}{\left(\frac{n+m}{2}\right)!\left(\frac{n-m}{2}\right)!}\cdot\frac{1}{2^n}$$

prob. of having n_h numbers of heads in n tosses

So

$$P(n, m) = \begin{cases} \frac{n!}{\left(\frac{n+m}{2}\right)!\left(\frac{n-m}{2}\right)!}\,\frac{1}{2^n}, & \text{if } n, m \text{ have the same parity;} \\ 0, & \text{if } n, m \text{ have different parity.} \end{cases}$$



Average position after n tosses:

$$\sum_{m=-n}^{n} m\cdot P(n,m) = \frac{n!}{2^n}\cdot\sum_{m=-n}^{n}\frac{m}{\left(\frac{n+m}{2}\right)!\left(\frac{n-m}{2}\right)!} =$$

$$= \frac{n!}{2^n}\left[\sum_{m=1}^{n}\frac{m}{\left(\frac{n+m}{2}\right)!\left(\frac{n-m}{2}\right)!} + \underbrace{\sum_{m=-n}^{-1}\frac{m}{\left(\frac{n+m}{2}\right)!\left(\frac{n-m}{2}\right)!}}_{\overset{m\to -m}{=}\ \sum_{m=1}^{n}\frac{-m}{\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!}}\right] = 0$$

№ 21.1.13

Recall Stirling's formula: $\ln n! \approx n\ln n - n$.
— for large n

For $n \gg m$, $n \gg 1$:

$$\ln P(n,m) = \ln n! - \ln\left(\frac{n+m}{2}\right)! - \ln\left(\frac{n-m}{2}\right)! - n\ln 2 \approx$$

$$\approx n\ln n - \cancel{n} - \frac{\cancel{n}+m}{2}\cdot\ln\underbrace{\left(\frac{n+m}{2}\right)}_{=\frac{n}{2}\left(1+\frac{m}{n}\right)} + \frac{\cancel{n}+\cancel{m}}{2} -$$

$$- \frac{n-m}{2}\cdot\ln\underbrace{\left(\frac{n-m}{2}\right)}_{=\frac{n}{2}\left(1-\frac{m}{n}\right)} + \frac{\cancel{n}-\cancel{m}}{2} - n\ln 2 =$$

$$= n\ln n - \frac{n+\cancel{m}}{2}\cdot\ln\frac{n}{2} - \frac{n+m}{2}\underbrace{\ln\left(1+\frac{m}{n}\right)}_{\approx\frac{m}{n}-\left(\frac{m}{n}\right)^2\cdot\frac{1}{2}} - \frac{n-\cancel{m}}{2}\cdot\ln\frac{n}{2} - \frac{n-m}{2}\underbrace{\ln\left(1-\frac{m}{n}\right)}_{\approx-\frac{m}{n}-\left(\frac{m}{n}\right)^2\cdot\frac{1}{2}} - n\ln 2$$

$$\approx \overbrace{n\ln n - n\ln\frac{n}{2} - n\ln 2}^{=0} \cancel{- \frac{m}{2} + \frac{m^2}{2n} + \frac{m^2}{4n}}$$

$$\cancel{+ \frac{m}{2} - \frac{m^2}{2n} + \frac{m^2}{4n} = -\frac{m^2}{2n}}$$

Hence $P(n,m) \approx e^{-\frac{m^2}{2n}}$