Tutorial #1

(homogen. : RHS = 0)

(1.10) $u'''_{(x)} - 3u''_{(x)} + 4u_{(x)} = 0$ - linear ✓ ODE with const coeff.

^ sometimes may write t [import. since often PDE → ODE]

Subs $u = e^{\lambda x}$ ⟹ Charact. eq: (by cancel exp that can't attain 0)

$$\lambda^3 - 3\lambda^2 + 4 = 0$$

Guess $\lambda_1 = -1$ - single root. Let's find $\lambda_{2,3}$.

$$\begin{array}{l|l} \lambda^3 - 3\lambda^2 + 4 & \lambda + 1 \\ \hline -(\lambda^3 + \lambda^2) & \underbrace{\lambda^2 - 4\lambda + 4}_{=(\lambda-2)^2} \\ \quad -4\lambda^2 + 4 & \\ \quad -(-4\lambda^2 - 4\lambda) & \\ \qquad 4\lambda + 4 & \\ \qquad -(4\lambda + 4) & \\ \qquad\quad 0 & \end{array}$$

⟹ Resulting factorization:

$$\lambda^3 - 3\lambda^2 + 4 = (\lambda+1)(\lambda-2)^2 = (\lambda+1)(\lambda^2 - 4\lambda + 4)$$

$\lambda_{2,3} = 2$ - multiple root, i.e. $\lambda_2 = \lambda_3$

Recall: In general, sol. to this type eq. if $\lambda_1, \dots, \lambda_{n-1}$ - real differ. roots, λ_n - real root of mult. m

$$u(x) = \underbrace{\sum_{k=1}^{n-1} C_k e^{\lambda_k x}}_{\text{obvious by superposition (valid due to linearity)}} + \left(\sum_{k=0}^{m-1} d_k x^k\right) \cdot e^{\lambda_n x}$$

In our case:

$$u(x) = C_1 e^{-x} + \underbrace{C_2}_{=d_0} e^{2x} + \underbrace{C_3}_{=d_1} x e^{2x}$$ - superposition of sols $\{e^{-x}, e^{2x}, x e^{2x}\}$

form basis of vect. (3D) space, since all 3 sols are lin. indep. & ODE is of order 3 also.

1.5.2

$$\begin{cases} u''(x) + u'(x) = f(x) \\ u'(0) = u(0) = \frac{1}{2}\left[u'(l) + u(l)\right] \end{cases} \quad (*)$$

Solution : $\underbrace{\text{uniqueness + existence}}_{\text{part of well-posedness}}$?

a) Uniqueness:

Let $u(x), v(x)$ be sol-s to $(*)$.

Define $w(x) = u(x) - v(x)$

$$\begin{cases} w''(x) + w'(x) = 0 \\ w(0) = \frac{1}{2}\left(w'(l) + w(l)\right) \\ w'(0) = \frac{1}{2}\left(w'(l) + w(l)\right) \end{cases}$$

— does it have non-triv. sol. ?

(if trivial, $w(x) \equiv 0 \Rightarrow u(x) = v(x)$)

uniqueness

$$w''(x) + w'(x) = 0 \Rightarrow w'(x) + w(x) = C_1$$

$$\Rightarrow w(x) = \underbrace{C_2 e^{-x}}_{\text{gen. homog. eq. sol.}} + \underbrace{C_1}_{\text{part. non-homog. eq. sol.}}$$

summing up a orig. cond.

$$\begin{cases} w(0) + w'(0) = w'(l) + w(l) \\ w(0) = w'(0) \end{cases} \Rightarrow \begin{cases} \cancel{C_2} + C_1 - \cancel{C_2} = \cancel{C_2 e^{-l}} + C_1 - \cancel{C_2 e^{-l}} \\ C_2 + C_1 = -C_2 \Rightarrow C_1 = -2C_2 \end{cases}$$

orig. cond.

automatically satisfied

$$w(x) = C_2\left(e^{-x} - 2\right)$$ arbitr. const.

Sol. to $(*)$ is non-unique

1.5.2 b) Existence :

Let's just solve the problem

gener. antiderivative

antiderivative

$$u''(x) + u'(x) = f(x) \Rightarrow u'(x) + u(x) = \cancel{\int_{x_1}^{x} f(s)\,ds} = F(x) + C_1, \quad (\Delta)$$

where $F'(x) = f(x)$

$$u'(x) + u(x) = 0 \Rightarrow u(x) = C \cdot e^{-x}$$

$$u'(x) + u(x) = F(x) + C_1 \Rightarrow u(x) = C(x) \cdot e^{-x}$$

To find $C(x)$, plug this back to the eq.: — by method of variation of arbitrary constants

$$C'(x) \cdot e^{-x} = F(x) + C_1 \Rightarrow C(x) = \int_{x_0}^{x} F(s) \cdot e^{s}\,ds + C_1 e^{x} \ominus$$

convenient to use defin. integr. with undef. x_0 instead of antideriv.

$$\ominus\ F(s) \cdot e^{s} \Big|_{x_0}^{x} - \int_{x_0}^{x} \underbrace{F'(s)}_{=f(s)} e^{s}\,ds + C_1 e^{x} =$$

$$= F(x) \cdot e^{x} - \int_{x_0}^{x} f(s)\,e^{s}\,ds + C_1 e^{x} - F(x_0) \cdot e^{x_0}$$

$$\Rightarrow u(x) = F(x) - e^{-x} \int_{x_0}^{x} f(s)\,e^{s}\,ds - F(x_0)\,e^{x_0 - x} + C_1$$

$$u'(x) = \cancel{f(x)} + e^{-x} \int_{x_0}^{x} f(s)\,e^{s}\,ds \cancel{- f(x)} + F(x_0) \cdot e^{x_0 - x}$$

B.C.:

$$\left.\begin{aligned} u(0) &= \int_{x_0}^{0} f(s)\,e^{s}\,ds + F(x_0)\,e^{x_0} \\ u'(0) &= F(0) - \int_{x_0}^{0} f(s)\,e^{s}\,ds + C_1 - F(x_0) \cdot e^{x_0} \end{aligned}\right) \Rightarrow C_1 = 2\int_{x_0}^{0} f(s)\,e^{s}\,ds + 2F(x_0)\,e^{x_0} - F(0)$$

$$\tfrac{1}{2}\left[u'(\ell) + u(\ell)\right] \overset{\text{by } (\Delta)}{=} \tfrac{1}{2}\left(F(\ell) + C_1\right) = u'(0) = \int_{x_0}^{0} f(s)\,e^{s}\,ds + F(x_0)\,e^{x_0}$$

$$\Rightarrow F(\ell) - F(0) + 2\int_{x_0}^{0} f(s)e^{s}ds + 2F(x_0)e^{x_0} = 2\int_{x_0}^{0} f(s)e^{s}ds + F(x_0)e^{x_0}$$

$$\Rightarrow F(\ell) = F(0)$$

In other words,

$$F(\ell) - F(0) = \int_{x_1}^{\ell} f(s)\,ds - \underbrace{\int_{x_1}^{0} f(s)ds}_{= -\int_0^{x_1} f(s)ds} = \int_0^{\ell} f(s)\,ds = 0, \text{ i.e.}$$

$\langle f \rangle := \frac{1}{\ell}\int_0^{\ell} f(s)ds =$

Hence, the sol. exist only if $\langle f \rangle = 0$

1.5.1 $\begin{cases} y'' + y = 0 \\ y(0) = 0,\ y(L) = 0 \end{cases}$ - is trivial solution unique

gen. sol.

$y'' + y = 0 \Rightarrow y(x) = C_1 \sin x + C_2 \cos x$

$y(0) = 0 \Rightarrow C_2 = 0$

$y(L) = 0$

$\Rightarrow y(L) = C_1 \cdot \sin L = 0$

for general L

$\Leftarrow C_1 = 0$ ~~for~~

$\Leftarrow y(x) = 0$ - unique

Not if $L = n \cdot \pi,\ n \in \mathbb{Z}$

$y(L) = C_1 \cdot \underbrace{\sin(n\pi)}_{=0} = 0 \xRightarrow{\cancel{for}} y(x) = C_1 \cdot \sin x$ - sol. for $\forall C_1$

($\exists$ ∞-many of them, $y \equiv 0$ is just a particular one)