Tutorial #10

N 5.6.4

$$\begin{cases} u_{tt} = c^2 u_{xx} + k, & 0<x<l \\ u(0,t)=0, \ u_x(l,t)=0 \\ u(x,0)=0, \ u_t(x,0)=V \end{cases}$$

Use method of shifting the data to reduce the problem to a homogeneous one.

Set $v_{(x,t)} = u_{(x,t)} - \overbrace{(ax^2+bx+d)}^{\text{shifting function}}$

a, b, d — const. to be found

this is a general form (w/o t-dependence that would spoil "BC") of suitable "shift" since $\left(\partial^2_{tt} - c^2\partial_{xx}\right)(ax^2+bx+d) = \text{const}$ so can compensate k in the original eq.

We require:

$$v_{tt} = c^2 v_{xx} \Rightarrow u_{tt} = c^2\left[u_{xx} - 2a\right]$$

$$\Rightarrow -2ac^2 = k \Rightarrow a = -\frac{k}{2c^2}$$

Also want BC to remain homogeneous:

$$\begin{cases} v(0,t)=0 \\ v_x(l,t)=0 \end{cases} \Rightarrow \begin{array}{l} v(0,t) = \underbrace{u(0,t)}_{=0} - d = 0 \Rightarrow d=0 \\ v_x(l,t) = \underbrace{u_x(l,t)}_{=0} - (2al+b) = 0 \Rightarrow b = -2al = \\ \qquad = \frac{kl}{c^2} \end{array}$$

Hence, we have: $v(x,t) = u(x,t) - \left(-\frac{k}{2c^2}x^2 + \frac{Kl}{c^2}x\right) = u(x,t) + \frac{k}{c^2}x\left(\frac{x}{2} - l\right)$

$$\begin{cases} v_{tt} = c^2 v_{xx} \\ v(0,t) = 0, \quad v_x(l,t) = 0 \\ v(x,0) = \frac{kx}{c^2}\cdot\left(x/2 - l\right), \quad v_t(x,0) = V \end{cases}$$

$v(x,t) = X(x)\cdot T(t) \Longrightarrow$

$$\frac{T''}{c^2\cdot T} = \frac{X''}{X} = -\lambda \Longrightarrow X'' + \lambda X = 0$$

$X(x) = C_1 \sin(\sqrt{\lambda}\,x) + C_2\cos(\sqrt{\lambda}\,x)$

$X(0) = 0 \Rightarrow C_2 = 0$

$X'(l) = 0 \Rightarrow \cos(\sqrt{\lambda}\,l) = 0$

$\Rightarrow \sqrt{\lambda}\,l = \pi\cdot(n + 1/2)$

$\lambda_n = \frac{\pi^2(2n+1)^2}{4l^2}$

$X_n(x) = \sin\left(\frac{\pi(2n+1)}{2l}x\right)$

$T'' + \lambda c^2 T = 0$

$$T(t) = A_n\cdot\cos\left(\frac{\pi(2n+1)ct}{2l}\right) + B_n\cdot\sin\left(\frac{\pi(2n+1)ct}{2l}\right)$$

$$v(x,t) = \sum_{n=0}^{\infty}\left(A_n\cdot\cos\left(\frac{\pi(2n+1)}{2l}ct\right) + B_n\cdot\sin\left(\frac{\pi(2n+1)}{2l}ct\right)\right)\cdot\sin\left(\frac{\pi(2n+1)}{2l}x\right)$$

$$v(x,0) = \sum_{n=0}^{\infty} A_n\cdot\sin\left(\frac{\pi(2n+1)}{2l}x\right) = Kx\left(x/2 - l\right)$$

Let's expand: $\frac{Kx}{c^2}\left(x/2 - l\right) = \sum_{m=1}^{\infty} a_m \cdot \sin\left(\frac{\pi m x}{2l}\right)$ on $x\in(0, 2l)$

$$a_m = \frac{k}{lc^2}\int_0^{2l} x\left(x/2 - l\right)\cdot\sin\left(\frac{\pi m}{2l}x\right)dx = \frac{k}{lc^2}\cdot\frac{2l}{\pi m}\int_0^{2l}(x-l)\cdot\cos\left(\frac{\pi m x}{2l}\right)dx =$$

$$= -\frac{k}{lc^2}\left(\frac{2l}{\pi m}\right)^2\cdot\int_0^{2l}\sin\left(\frac{\pi m}{2l}x\right)dx = \frac{k}{lc^2}\cdot\left(\frac{2l}{\pi m}\right)^3\cdot\left[\underbrace{\cos(\pi m)}_{=(-1)^m} - 1\right]$$

Notice: $a_m = 0$ for m-even, whereas

for m-odd: $a_m = a_{2n+1} = A_n = -\frac{2K}{c^2 l}\left[\frac{2l}{\pi(2n+1)}\right]^3$

(put $m = 2n+1$)

$$v_t(x,0) = \frac{\pi c}{2l} \cdot \sum_{n=0}^{\infty} B_n \cdot (2n+1) \cdot \sin\left(\frac{\pi(2n+1)x}{2l}\right) = V$$

As before, expand:

$$V = \sum_{m=1}^{\infty} b_m \sin\left(\frac{\pi m}{2l} x\right) \quad \text{on} \quad x \in (0, 2l)$$

$$b_m = \frac{V}{l} \cdot \int_0^{2l} \sin\left(\frac{\pi m}{2l} x\right) dx = -\frac{V}{l} \cdot \frac{2l}{\pi m}\Big[\underbrace{\cos(\pi m)}_{=(-1)^m} - 1\Big];$$

Matching with the above series:

$$b_{2n+1} = B_n \cdot (2n+1) = \frac{4V}{\pi \cdot (2n+1)} \Rightarrow$$

$$\Rightarrow B_n = \frac{4V}{\pi (2n+1)^2}$$

Finally,

$$u(x,t) = Kx\left(l - x/2\right) + \sum_{n=0}^{\infty}\left(-\frac{2K}{c^2 l}\left[\frac{2l}{\pi(2n+1)}\right]^3 \cdot \cos\left(\frac{\pi(2n+1)c}{2l} t\right) + \right.$$

$$\left. + \frac{4V}{\pi(2n+1)^2} \cdot \sin\left(\frac{\pi(2n+1)c}{2l} t\right)\right) \cdot \sin\left(\frac{\pi(2n+1)}{2l} x\right)$$

N 6.1.4

$$\begin{cases} \Delta u = 0 \\ u\big|_{r=a} = A, \quad u\big|_{r=b} = B \end{cases}$$

depend only on r $\Rightarrow$ $u(r, \theta, \varphi) = u(r)$

Laplacian in spherical coordinates

$$\Delta u = \underbrace{u_{rr}}_{= u''} + \frac{2}{r}\underbrace{u_r}_{= u'} + 0 = 0$$

Let $u = v/r$

$$u' = v'/r - v/r^2$$

$$u'' = v''/r - 2v'/r^2 + 2v/r^3$$

$$u'' + \frac{2}{r}u' = v''/r \cancel{- 2v'/r^2} \cancel{+ 2v/r^3} \cancel{+ 2v'/r^2} \cancel{- \frac{2v}{r^3}} = 0$$

$$\Rightarrow v'' = 0 \Rightarrow v(r) = C_1 \cdot r + C_2$$

$$v(a)/a = A, \quad v(b)/b = B:$$

$$\begin{cases} C_1 \cdot a + C_2 = a \cdot A \\ C_1 \cdot b + C_2 = b \cdot B \end{cases} \Rightarrow \begin{cases} C_1 = \dfrac{a \cdot A - b \cdot B}{a - b} \\ C_2 = \dfrac{ab \cdot (A - B)}{b - a} \end{cases}$$