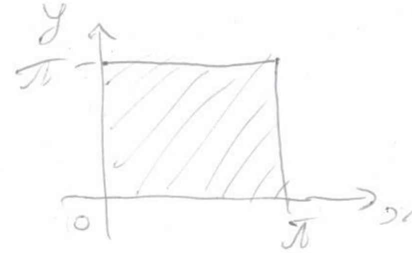
Tutorial #11

№ 6.2.3



$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u_y\big|_{y=0} = 0, \quad u_y\big|_{y=\pi} = 0 \\ u\big|_{x=0} = 0, \quad u\big|_{x=\pi} = \cos^2 y = \frac{1}{2}(1 + \cos 2y) \end{cases}$$

Use separation of variables (domain is simple!):

$$u(x,y) = X(x) \cdot Y(y) \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

$$\begin{cases} Y'' + \lambda Y = 0 \\ Y'(0) = 0, \quad Y'(\pi) = 0 \end{cases} \Rightarrow Y_n(y) = \cos(ny), \quad \lambda_n = n^2$$

$$X'' - n^2 X = 0 \quad \xRightarrow{n>0} \quad X(x) = C_n \cdot e^{nx} + B_n \cdot e^{-nx}$$

$$\left(= \hat{C}_n \cdot \sinh(nx) + \hat{B}_n \cdot \cosh(nx) \right)$$

$$\xrightarrow{n=0} \quad X(x) = B_0 + C_0 \cdot x$$

$$u(x,y) = B_0 + C_0 x + \sum_{n=1}^{\infty} \left[C_n \cdot e^{nx} + B_n \cdot e^{-nx} \right] \cdot \cos(ny)$$

Left boundary:

$$u(0,y) = B_0 + \sum_{n=1}^{\infty} (C_n + B_n) \cdot \cos(ny) = 0 \Rightarrow \begin{cases} B_0 = 0 \\ B_n = -C_n \end{cases}$$

by lin. independence of $\{1, \cos(ny)\}_{n=1}^{\infty}$

$$u(x,y) = C_0 \cdot x + \sum_{n=1}^{\infty} C_n \underbrace{\left(e^{nx} - e^{-nx}\right)}_{= 2\sinh(nx)} \cos(ny)$$

Right boundary:

$$u(\pi, y) = C_0 \pi + 2 \sum_{n=1}^{\infty} C_n \cdot \sinh(n\pi) \cdot \cos(ny) = \frac{1}{2} + \frac{1}{2}\cos(2y)$$

$$\Rightarrow \begin{cases} C_0 = \frac{1}{2\pi} \\ C_2 = \frac{1}{4 \cdot \sinh(2\pi)} \\ C_n = 0 \quad \forall n \notin \{0, 2\} \end{cases} \Rightarrow u(x,y) = \frac{1}{2\pi} x + \frac{1}{2 \cdot \sinh(2\pi)} \cdot \sinh(2x) \cdot \cos(2y)$$

N6.3.2

$$\begin{cases} \Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \\ u\big|_{r=a} = 1 + 3\sin\theta \end{cases}$$

$$u(r,\theta) = R(r)\cdot \Psi(\theta) \Rightarrow r^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R}\right) = -\frac{\Psi''}{\Psi} = \lambda$$

$$\Rightarrow \begin{cases} \Psi'' + \lambda\Psi = 0 \\ r^2 R'' + rR' - \lambda R = 0 \end{cases}$$

$$\begin{cases} \Psi'' + \lambda\Psi = 0 \\ \Psi(\theta) = \Psi(\theta + 2\pi) \end{cases} \Rightarrow \Psi(\theta) = C\cdot\cos(\sqrt{\lambda}\cdot\theta) + D\cdot\sin(\sqrt{\lambda}\cdot\theta)$$

$$\Psi(\theta + 2\pi) = C\cdot\cos(\sqrt{\lambda}\theta + \sqrt{\lambda}\cdot 2\pi) + D\sin(\sqrt{\lambda}\theta + \sqrt{\lambda}\cdot 2\pi)$$

$$\Rightarrow \sqrt{\lambda} = n \Rightarrow \lambda_n = n^2$$

$$\Psi_n(\theta) = C_n\cdot\cos(n\theta) + D_n\cdot\sin(n\theta)$$

$$r^2 R'' + rR' - n^2 R = 0$$

$n=0$: $\underbrace{\frac{R''}{R'}}_{=(\log R')'} = -\frac{1}{r} \Rightarrow \log R' = -\log\left(\frac{r}{A}\right) \Rightarrow R' = A/r$

$R(r) = A\cdot\log r + B$

$n>0$: Sub $R(r) = r^\alpha$ for some α to be determined $\Rightarrow \left(\alpha(\alpha-1) + \alpha - n^2\right)\cancel{r^\alpha} = 0 \Rightarrow \alpha = \pm n$

$$u(r,\theta) = A_0\cdot\log r + B_0 + \sum_{n=1}^{\infty}\left(A_n r^n + B_n\cdot\frac{1}{r^n}\right)\left(C_n\cdot\cos(n\theta) + D_n\cdot\sin(n\theta)\right)$$

$u(0,\theta) < \infty$ - for physical reasons ($r=0$ is in our domain where we expect solution to be defined)

$\Rightarrow A_0 = 0$, $B_n = 0$ for $\forall n = 1, 2, \ldots$

Rename: $C_n\cdot A_n \to C_n$, $D_n\cdot A_n \to D_n$

Then the general solution is:

$$u(r,\theta) = B_0 + \sum_{n=1}^{\infty}\left[C_n \cdot \cos(n\theta) + D_n \cdot \sin(n\theta)\right] \cdot r^n$$

$$u(a,\theta) = 1 + 3\sin\theta \implies B_0 = 1, \quad D_1 = \frac{3}{a}, \quad D_n = 0 \text{ for } \forall n = 2, 3, \ldots$$

$$C_n = 0 \text{ for } \forall n = 1, 2, \ldots$$

Finally:

$$u(r,\theta) = 1 + 3 \cdot \sin\theta \cdot r/a$$

$\mathcal{N}$ 5.6.6

$$\begin{cases} u_{tt} = c^2 u_{xx} + g(x)\cdot\sin(\omega t) \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = u_t(x,0) = 0 \end{cases}$$

Set $v(x,t) = u(x,t) - A(x)\cdot\sin(\omega t)$

Compute: $v_{tt} = u_{tt} + \omega^2\cdot A(x)\cdot\sin(\omega t)$

$v_{xx} = u_{xx} - A''(x)\cdot\sin(\omega t)$

We want to have: $v_{tt} = c^2 v_{xx}$

$$\Rightarrow \underbrace{u_{tt}}_{= \cancel{c^2 u_{xx}} + g(x)\cdot\sin(\omega t)} + \omega^2 A(x)\sin(\omega t) = \cancel{c^2 u_{xx}} - c^2 A''(x)\cdot\sin(\omega t)$$

$$\Rightarrow \begin{cases} A''(x) + \frac{\omega^2}{c^2} A(x) = g(x)/c^2 \\ A(0) = A(l) = 0 \end{cases}$$

since we also want $v(0,t) = v(l,t) = 0$

Can solve using Green's function method

Green's function $G(x) \equiv G(x,y)$ (for $\forall y \in (0,l)$ fixed) solves:

$$\begin{cases} G'' + \omega^2/c^2\, G = \delta(x-y) \\ G(0) = 0,\ G(l) = 0 \end{cases}$$

For $x<y$: $\begin{cases} G'' + \omega^2/c^2\cdot G = 0 \\ G(0) = 0 \end{cases} \Rightarrow G_{<y}(x) = C_0\cdot\sin\left(\omega/c\cdot x\right)$

notation for solution in this case

For $x>y$: $\begin{cases} G'' + \omega^2/c^2\, G = 0 \\ G(l) = 0 \end{cases}$

$$\Rightarrow G_{>y}(x) = \tilde{C}_1\cdot\sin(\omega/c\, x) + \tilde{C}_2\cdot\cos(\omega/c\, x)$$

$$\tilde{C}_1\cdot\sin(\omega/c\, l) = -\tilde{C}_2\cdot\cos(\omega/c\cdot l)$$

$$\tilde{C}_2 = -\tilde{C}_1\cdot\frac{\sin(\omega/c\, l)}{\cos(\omega/c\, l)}$$

$$G_{>y}(x) = \frac{\tilde{C}_1}{\cos(\omega/c\, l)} \underbrace{\left[\sin(\omega/c\, x)\cdot\cos(\omega/c\, l) - \cos(\omega/c\, x)\cdot\sin(\omega/c\, l)\right]}_{=\sin[\omega/c(x-l)]}$$

At $x=y$: ① $G_{<y}(y) = G_{>y}(y)$ - continuity (natural condition)

② $G'_{>y}(y) - G'_{<y}(y) = 1$ - jump of derivative (condition dictated by the equation)

can be obtained from $\lim\limits_{\varepsilon\to 0}\int\limits_{y-\varepsilon}^{y+\varepsilon}$ both sides of ODE

①: $$\frac{\tilde{C}_1}{\cos(\omega/c\, l)} \underbrace{\left[\sin(\omega/c\, y)\cdot\cos(\omega/c\, l) - \cos(\omega/c\, y)\cdot\sin(\omega/c\, l)\right]}_{=\sin[\omega/c(y-l)]} = C_0\cdot\sin(\omega/c\, y)$$

$$\Rightarrow \tilde{C}_1 = C_0\cdot\frac{\sin(\omega/c\, y)\cdot\cos(\omega/c\, l)}{\sin[\omega/c(y-l)]}$$

②: $$\frac{\tilde{C}_1\cdot\omega/c}{\cos(\omega/c\, l)} \overbrace{\left[\cos(\omega/c\, y)\cdot\cos(\omega/c\, l) + \sin(\omega/c\, y)\cdot\sin(\omega/c\, l)\right]}^{=\cos[\omega/c(y-l)]} -$$

$$- C_0\cdot\omega/c\,\cos(\omega/c\, y) = 1$$

$$\Rightarrow C_0 = \frac{c/\omega\cdot\sin[\omega/c(y-l)]}{\sin(\omega/c\, y)\cdot\cos[\omega/c(y-l)] - \cos(\omega/c\, y)\cdot\sin[\omega/c(y-l)]} =$$

$$= \frac{c/\omega\cdot\sin[\omega/c(y-l)]}{\sin(\omega/c\, l)}$$

$$\tilde{C}_1 = \frac{c/\omega\cdot\sin(\omega/c\, y)\cdot\cos(\omega/c\, l)}{\sin(\omega/c\, l)}$$

$$G(x,y) = G(x) = \begin{cases} G_{<y}(x) = c/\omega\cdot\dfrac{\sin[\omega/c(y-l)]\cdot\sin(\omega/c\, x)}{\sin(\omega/c\, l)}, & x<y \\ G_{>y}(x) = c/\omega\cdot\dfrac{\sin(\omega/c\, y)}{\sin(\omega/c\, l)}\cdot\sin[\omega/c(x-l)], & x>y \end{cases}$$

$$A(x) = \int_0^l G(x,y) \cdot g(y)/_{c^2} \, dy =$$

$$= \frac{1}{\omega c} \cdot \left(\frac{\sin\left[\omega/_c (x-l)\right]}{\sin(\omega/_c l)} \cdot \int_0^x \sin(\omega/_c y) \cdot g(y)\, dy + \frac{\sin(\omega/_c x)}{\sin(\omega/_c l)} \cdot \int_x^l \sin\left[\omega/_c (y-l)\right] g(y)\, dy \right)$$

– can be evaluated for given $g(x)$.

$$\begin{cases} v_{tt} = c^2 v_{xx} \\ v(0,t) = v(l,t) = 0 \\ v(x,0) = 0, \quad v_t(x,0) = -\omega \cdot A(x) \end{cases} \Rightarrow v(x,t) = \sum_{n=1}^{\infty} \left[\underbrace{C_n}_{0} \cdot \cos\left(\frac{n\pi c t}{l}\right) + D_n \cdot \sin\left(\frac{n\pi c t}{l}\right) \right] \cdot \sin\left(\frac{n\pi x}{l}\right)$$

– from $v(x,0) = 0$

$$\Rightarrow D_n = -\frac{\omega \cdot l}{n\pi c} \cdot \frac{2}{l} \int_0^l A(x) \cdot \sin\left(\frac{n\pi x}{l}\right) dx$$

Finally:

$$u(x,t) = A(x) \cdot \sin(\omega t) + \overbrace{\sum_{n=1}^{\infty} D_n \cdot \sin\left(\frac{n\pi c t}{l}\right) \cdot \sin\left(\frac{n\pi x}{l}\right)}^{= v(x,t)},$$

where $D_n = -\dfrac{2\omega}{n\pi c} \cdot \displaystyle\int_0^l A(x) \cdot \sin\left(\frac{n\pi x}{l}\right) dx$

and $A(x)$ is given at the top of the page.

Notice if $\sin(\omega/_c l) = 0$, $A(x)$ blows up, this corresponds to resonance if frequency is very specific:

$$\omega/_c l = \pi n \Rightarrow \omega_n = \frac{\pi c}{l} n, \quad n = 1, 2, \ldots$$