1.2.8 Tutorial #2

$a u_x + b u_y + c u = 0$

Introduce $\xi = ax + by$

$\eta = bx - ay$

Then $\dfrac{\partial}{\partial x} = \underbrace{\dfrac{\partial \xi}{\partial y}}_{=a} \cdot \dfrac{\partial}{\partial \xi} + \underbrace{\dfrac{\partial \eta}{\partial x}}_{=b} \cdot \dfrac{\partial}{\partial \eta}$ $\quad u_x = a \cdot u_\xi + b u_\eta$

$\Rightarrow u_y = b u_\xi - a u_\eta$

$\dfrac{\partial}{\partial y} = \underbrace{\dfrac{\partial \xi}{\partial y}}_{=b} \cdot \dfrac{\partial}{\partial \xi} + \underbrace{\dfrac{\partial \eta}{\partial y}}_{=-a} \cdot \dfrac{\partial}{\partial \eta}$

Let's find also expressions for x, y in terms of ξ, η.

$$\begin{matrix} a\times \\ + \\ b\times \end{matrix} \begin{cases} \xi = ax + by \\ \eta = bx - ay \end{cases} \begin{matrix} \times b \\ - \\ \times a \end{matrix}$$

$x = \dfrac{a\xi + b\eta}{a^2 + b^2}$ $\qquad y = \dfrac{b\xi - a\eta}{a^2 + b^2}$

Plugging derivatives above into the original eq., we obtain

$(a^2 + b^2)\dfrac{\partial u(\xi,\eta)}{\partial \xi} = -c \cdot u(\xi,\eta) \xRightarrow{\text{integrating w.r.t. } \xi} u(\xi,\eta) = \underbrace{f(\eta)}_{\text{const. w.r.t. } \xi} \cdot \exp\left(-\dfrac{c}{a^2+b^2}\xi\right)$

Getting back to the original variables x, y:

$u(x,y) = f(b x - ay) \cdot \exp\left[-\dfrac{c}{a^2+b^2}(ax + by)\right]$

1.2.7

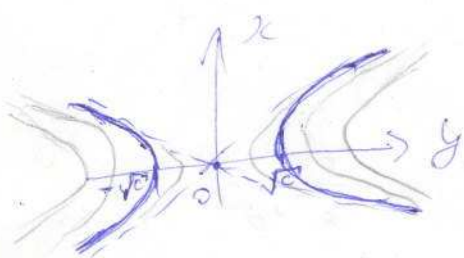
$$\begin{cases} y u_x + x u_y = 0 \\ u(0,y) = e^{-y^2} \end{cases}$$

$$u_x + \frac{x}{y} u_y = 0$$

$$\Rightarrow \frac{d u(x, y(x))}{dx} = 0 \quad \text{if} \quad \frac{dy}{dx} = \frac{x}{y} \Rightarrow x^2 - y^2 = -c$$

$$\Downarrow \quad dy^2 = dx^2$$

Hence $u(x,y) = \text{const}$ along lines $x^2 - y^2 = -c$

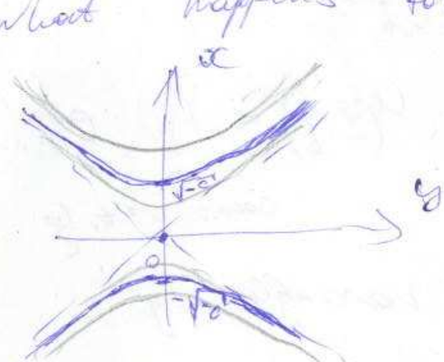


$$\Rightarrow u(x,y) = f(x^2 - y^2)$$

$$u(0,y) = e^{-y^2} \Rightarrow u(x,y) = e^{x^2 - y^2}$$

This is true only if $c > 0$, i.e. $x^2 - y^2 \le 0$, $x^2 \le y^2$

What happens for $c < 0$?



- no intersection with line $x = 0$ where initial data are prescribed: can't propagate $u(0,y) \to u(x,y)$

14.1.12

$$\begin{cases} u_t + u u_x = 1 \\ u(x,0) = x \end{cases}$$

Parametrize solution:

$$u(x,t) = u(s) = u(x(s), t(s))$$

$$u_t + u u_x = 1 \Rightarrow \begin{cases} \frac{du}{ds} = 1 \\ \frac{dt}{ds} = 1 \\ \frac{dx}{ds} = u \end{cases} \Rightarrow \begin{array}{l} u(s) = s + c_1 \\ t(s) = s + c_2 \\ x(s) = \frac{1}{2}(s+c_1)^2 + c_3 \end{array}$$

Initial data:

$$t(0) = 0 \Rightarrow c_2 = 0 \Rightarrow t(s) = s$$

$$u(0) = c_1 = x(0) = \frac{1}{2} c_1^2 + c_3 \Rightarrow c_3 = c_1 - \frac{1}{2} c_1^2$$

$$\Rightarrow x(s) = \frac{1}{2}(s + c_1)^2 + c_1 - \frac{1}{2} c_1^2 = \frac{1}{2} s^2 + c_1(1+s)$$

$$\Rightarrow c_1 = \frac{x - s^2/2}{1+s} = \frac{x - t^2/2}{1+t}$$

$$u(x,t) = t + c_1 = \frac{x + t^2/2 + t}{1+t}$$