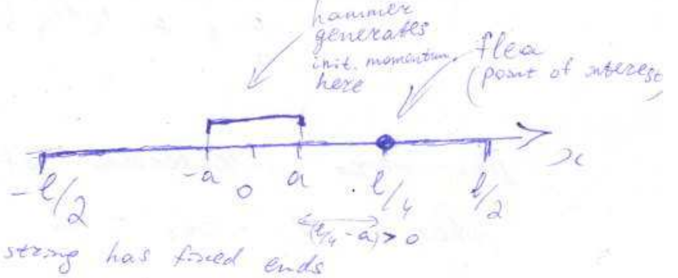
Tutorial #3

№ 2.1.3



$$\begin{cases} u_{tt} = \underbrace{\frac{T}{\rho}}_{=:c^2} \cdot u_{xx} \\ u(-\ell/2, t) = u(\ell/2, t) = 0 \quad \text{- since piano string has fixed ends} \\ u(x,0) = \varphi(x) \equiv 0 \\ u_t(x,0) = \psi(x) = \begin{cases} B, & |x| \le a \\ 0, & |x| > a \end{cases} \end{cases}$$

B – const. characterizing init momentum from hammer

Clearly, the time when the initial disturbance reaches the flea $(x = \ell/4)$ is smaller than time when it reaches an endpoint of the string $(x = \pm \ell/2)$. Within such time interval the boundary conditions $(u(\pm \ell/2, t) = 0)$ are automatically satisfied, so we can drop them and use model of infinite string.

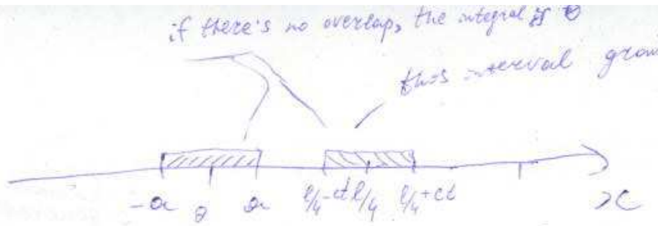
Thus, the solution for the time of interest is given by D'Alembert formula:

$$u(x,t) = \underbrace{\frac{\overset{=0}{\varphi}(x-ct) + \overset{=0}{\varphi}(x+ct)}{2}}_{=0} + \frac{1}{2c} \int_{x-ct}^{x+ct} \underbrace{\psi(\xi)}_{\text{either } = B \text{ or } 0} d\xi$$

We're interested in minimal time t_0 s.t.

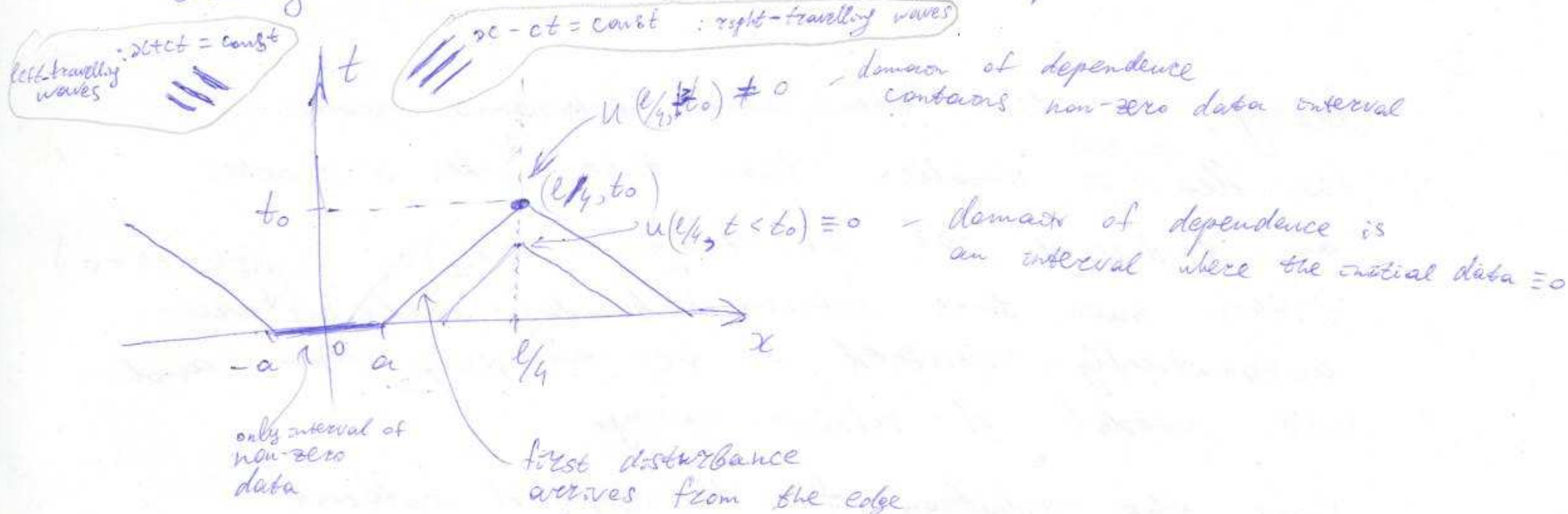
$$u(\ell/4, t_0) \neq 0$$

$$u(\ell/4, t) = \frac{B}{2c} \cdot \Big| \underbrace{(\ell/4 - ct, \ell/4 + ct) \cap (-a, a)}_{\text{length of the resulting (after intersection) interval}} \Big|$$



Non-zero intersection of these intervals happens when $l/4 - ct_0 = a$ i.e. at $t_0 = \frac{1}{c}\left(l/4 - a\right) = \sqrt{\frac{\rho}{T}} \cdot \left(l/4 - a\right)$

The same result can be obtained by looking at characteristic lines in plane $(t\, x)$:



first disturbance arrives from the edge of init. disturbance interval; the characteristic along which it propagates is : $t = 1/c\,(x-a)$

$\Rightarrow t_0 = \frac{1}{c}\left(l/4 - a\right)$

substituting point $(l/4, t_0)$

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$$\begin{cases} u_{xx} - 3u_{xt} - 4u_{tt} = 0 \\ u(x,0) = x^2 \\ u_t(x,0) = e^x \end{cases}$$

Let's factorize differentiation operator:

$$\left(\frac{\partial}{\partial x} + a\cdot\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + b\cdot\frac{\partial}{\partial t}\right) = \frac{\partial^2}{\partial x^2} + ab\frac{\partial^2}{\partial t^2} + (a+b)\underbrace{\frac{\partial^2}{\partial x\,\partial t}}_{=\frac{\partial^2}{\partial t\,\partial x}}$$

We want:

$$\begin{cases} a\cdot b = -4 \\ a+b = -3 \end{cases} \Rightarrow \left[\begin{array}{l} \begin{cases} a = 1 \\ b = -4 \end{cases} \text{ — take this w.l.o.g.} \\ \begin{cases} a = -4 \\ b = 1 \end{cases} \end{array}\right.$$

Hence, $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)u = 0$

Let's find a linear transformation: $(x,t) \to (\xi,\eta)$ s.t.:

$$\begin{cases} \frac{\partial}{\partial \xi} = \underbrace{1}_{=\frac{\partial x}{\partial \xi}}\cdot\frac{\partial}{\partial x} + \underbrace{(-4)}_{=\frac{\partial t}{\partial \xi}}\cdot\frac{\partial}{\partial t} \\ \frac{\partial}{\partial \eta} = \underbrace{1}_{=\frac{\partial x}{\partial \eta}}\cdot\frac{\partial}{\partial x} + \underbrace{1}_{=\frac{\partial t}{\partial \eta}}\cdot\frac{\partial}{\partial t} \end{cases} \Rightarrow \begin{cases} x = \xi + \eta \\ t = -4\xi + \eta \end{cases} \Rightarrow \begin{cases} \eta = \frac{1}{5}(4x+t) \\ \xi = \frac{1}{5}(x-t) \end{cases}$$

Then the PDE becomes:

$$u_{\xi\eta} = 0 \Rightarrow u_\xi = F(\xi) \Rightarrow u(\xi,\eta) = \underbrace{\int F(\xi)\,d\xi}_{=:f_0(\xi)} + g_0(\eta)$$

Back to the original coordinates:

$$u(x,t) = \underbrace{f_0\left(\tfrac{1}{5}(x-t)\right)}_{=f(x-t)} + \underbrace{g_0\left(\tfrac{1}{5}(4x+t)\right)}_{=g(4x+t)}$$

$$u_t(x,t) = -f'(x-t) + g'(4x+t)$$

Init. conditions:

$$\begin{cases} u(x,0) = x^2 = f(x) + g(4x) \\ u_t(x,0) = e^x = -f'(x) + g'(4x) \end{cases}$$

$\int \dots dx$ $\quad e^x = -f(x) + \frac{1}{4} g(4x) + C$

$$\Rightarrow \begin{cases} g(4x) = \frac{4}{5}\left(x^2 + e^x - C\right) \\ f(x) = \frac{1}{5}\left(x^2 - 4e^x + 4C\right) \end{cases}$$

$$\Rightarrow \begin{cases} f(x) = \frac{1}{5}\left(x^2 - e^x + 4C\right) \\ g(x) = \frac{4}{5}\left(\frac{1}{16}x^2 + e^{x/4} - C\right) \end{cases}$$

$$u(x,t) = \frac{1}{5}\left((x-t)^2 - 4e^{x-t}\right) + \frac{4}{5}\left(\frac{1}{16}(4x+t)^2 + e^{x+t/4}\right) \cancel{+\tfrac{4}{5}C} \cancel{-\tfrac{4}{5}C} =$$

$$= \frac{4}{5}\left(e^{x+t/4} - e^{x-t}\right) + \frac{1}{5}\left[x^2 - 2tx + t^2 + 4x^2 + 2tx + \frac{t^2}{4}\right] =$$

$$= \frac{4}{5}\left(e^{x+t/4} - e^{x-t}\right) + x^2 + \frac{t^2}{4}$$