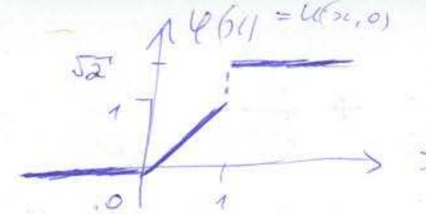
(N 1) - extra problem



$$\begin{cases} u_t + u^2 u_x = 0 \\ u(x,0) = \varphi(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ \sqrt{2}, & x > 1 \end{cases} \end{cases}$$

$$\begin{cases} \frac{dx(t)}{dt} = u^2(x(t),t) \\ \frac{du(x(t),t)}{dt} = 0 \end{cases} \Rightarrow \begin{cases} \frac{dx}{dt} = u^2(\underbrace{x(0)}_{=x_0},0) = \varphi^2(x_0) \\ u(x(t),t) = u(x_0,0) = \varphi(x_0) \end{cases}$$

$\Rightarrow \quad x = x_0 + \varphi^2(x_0) \cdot t$ - need to solve w.r.t. x_0

Since $\varphi(x)$ is defined piecewisely, consider 3 cases:

1) $x_0 < 0$: $\varphi(x_0) = 0 \Rightarrow x = x_0 + 0 \cdot t = x_0$

$u(x,t) = \varphi(x_0) = 0$ for $x = x_0 < 0$

2) $0 \le x_0 \le 1$: $\varphi(x_0) = x_0 \Rightarrow x = x_0 + x_0^2 \cdot t$

$\Rightarrow t x_0^2 + x_0 - x = 0$

$x_0 = \frac{-1 \mp \sqrt{1+4tx}}{2t}$

If take "$-$" root, we have $x_0 = \frac{-1-\sqrt{1+4tx}}{2t} < 0$

which is in contradiction with $0 \le x_0 \le 1$ (i.e. the case we consider)

Hence, $x_0 = \frac{-1+\sqrt{1+4tx}}{2t}$

$u(x,t) = \varphi(x_0) = x_0 = \frac{-1+\sqrt{1+4tx}}{2t}$, but where in (x,t)-plane is it valid? $1+4tx \ge 0 \Rightarrow x \ge -\frac{1}{4t}$;

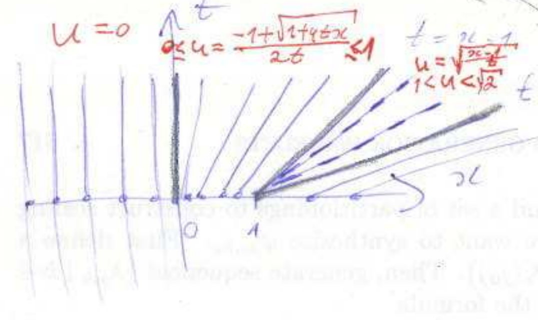
$x_0 \ge 0 \Rightarrow t \cdot x \ge 0 \Rightarrow x \ge 0$; $x_0 \le 1 \Rightarrow 1+4tx \le (2t+1)^2 \Rightarrow x \le \frac{\overbrace{(2t+1)^2 - 1}^{=2t\cdot 2(t+1)}}{4t} = t+1$

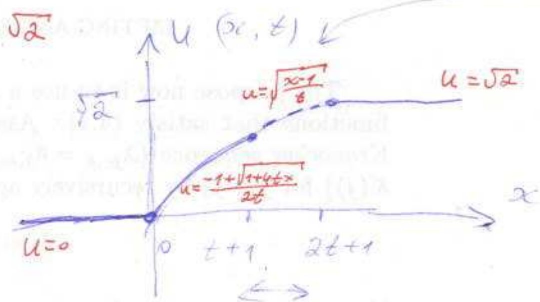
3) $x_0 > 1$: $\varphi(x_0) = \sqrt{2} \Rightarrow x = x_0 + 2t$

$\Rightarrow x_0 = x - 2t$

$u(x,t) = \varphi(x_0) = \sqrt{2}$ for $x > 2t+1$ (since $x_0 > 1$)

$t = \frac{x-1}{u^2}$ — do manual continuation in this region; this corresponds to extending solution continuously into the gap





Finally, conclude

$$u(x,t) = \begin{cases} 0, & x < 0 \\ \dfrac{-1+\sqrt{1+4tx}}{2t}, & 0 \le x \le t+1 \\ \sqrt{\dfrac{x-1}{t}}, & t+1 < x < 2t+1 \\ \sqrt{2}, & x \ge 2t+1 \end{cases}$$

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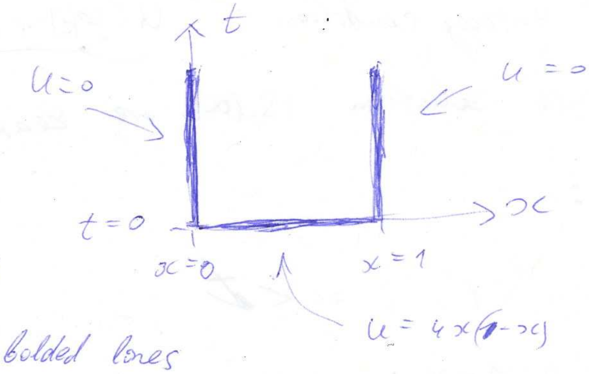
$$(*) \begin{cases} u_t = u_{xx}, & 0 < x < 1,\ 0 < t < \infty & (1) \\ u(0,t) = u(1,t) = 0, & 0 < t < \infty & (2) \\ u(x,0) = 4x(1-x) \equiv \varphi(x), & 0 \le x < 1 & (3) \end{cases}$$

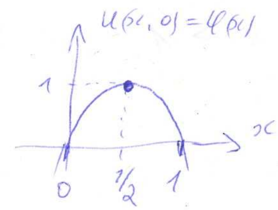
a) $\min u(x,t) \le u(x,t) \le \max u(x,t)$



According to the strong maximum principle min / max of $u(x,t)$ can be attained only somewhere on the bolded lines

(assuming $u(x,t) \not\equiv const$)

$\max\limits_{0\le x\le 1} u(x,0) = 1$



$\min\limits_{\substack{0\le x\le 1\\ t\ge 0}} u(x,t) = \min\limits_{0\le x\le 1} u(x,0) =$

$= \min\limits_{t>0} u(0,t) = \min\limits_{t>0} u(1,t) = 0$

Hence for $0<x<1$, $t>0$: $0 < u(x,t) < 1$

b) Introduce $v(x,t) := u(1-x,t)$. Then $v_t(x,t) = u_t(1-x,t)$

$v_x(x,t) = -u_x(1-x,t)$

$v_{xx}(x,t) = u_{xx}(1-x,t)$

$v_t - v_{xx} = u_t - u_{xx} = 0 \implies v_t = v_{xx}$ (1')

$v(0,t) = u(1,t) = 0$

$v(1,t) = u(0,t) = 0$ (2')

$v(x,0) = u(1-x,0) = 4(1-x)\cdot x = 4x\cdot(1-x) = u(x,0)$ (3')

Therefore, comparing (1)-(3) to (1')-(3') conclude that $v(x,t)$ satisfies (*). But, by uniqueness of solution we must have $\underbrace{v(x,t)}_{=u(1-x,t)} = u(x,t) \implies u(x,t) = u(1-x,t)$

c) $\int_0^1 u\cdot\,|\, u_t = u_{xx}\,|\cdot dx \implies \int_0^1 \frac{1}{2}(u^2)_t\,dx = \underbrace{\int_0^1 (u\cdot u_x)_x\,dx}_{= uu_x\big|^{x=1} = 0} - \underbrace{\int_0^1 (u_x)^2\,dx}_{>0,\ \text{otherwise if } u_x\equiv 0 \implies u(x,t)=U(t),} < 0$

$\implies \frac{d}{dt}\left[\int_0^1 u^2(x,t)\,dx\right] < 0$

but u(x,t) obviously depends on x