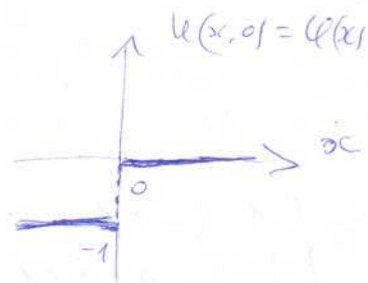
(№2) - extra problem

$$\begin{cases} u_t + u u_x = 1 \\ u(x,0) = \varphi(x) = \begin{cases} -1, & x < 0 \\ 0, & x \geq 0 \end{cases} \end{cases}$$



$$\begin{cases} \dfrac{dx(t)}{dt} = u(x(t),t) \\ \dfrac{d\,u(x(t),t)}{dt} = 1 \end{cases} \Rightarrow \begin{cases} x(t) = \frac{1}{2}t^2 + \varphi(x_0)\,t + x_0 \\ u(x(t),t) = t + \underbrace{\varphi(x_0)}_{=u(x(0),\,0),\ x(0)=x_0} \end{cases}$$

$$\cancel{x(t)}\quad x = \frac{t^2}{2} + \varphi(x_0)\cdot t + x_0$$

1) $x_0 < 0$: $\varphi(x_0) = -1 \Rightarrow x = \frac{t^2}{2} - t + x_0$

$\Rightarrow x_0 = x + t - \frac{t^2}{2}$; $u(x,t) = 2t + x - \frac{t^2}{2}$

$x_0 < 0 \Rightarrow x < \frac{t^2}{2} - t$

2) $x_0 \geq 0$: $\varphi(x_0) = 0 \Rightarrow x = \frac{t^2}{2} + x_0$

$\Rightarrow x_0 = x - \frac{t^2}{2}$, $u(x,t) = t$

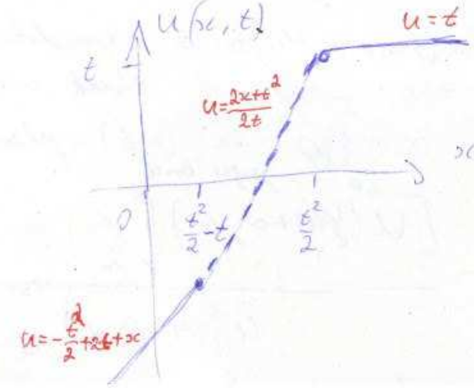
$x_0 > 0 \Rightarrow x > t^2/2$

Continuously filling the gap using $\begin{cases} u = t + \varphi(0) \\ x = \frac{t^2}{2} + \varphi(0)\cdot t \end{cases}$ $\Leftrightarrow$ $x = \frac{t^2}{2} + (u-t)\cdot t$

by formally eliminating $\varphi(0)$

Finally:

$$u(x,t) = \begin{cases} -\frac{t^2}{2} + 2t + x, & x < \frac{t^2}{2} - t \\ \dfrac{2x+t^2}{2t}, & \frac{t^2}{2} - t \leq x < \frac{t^2}{2} \\ t, & x \geq t^2/2 \end{cases}$$



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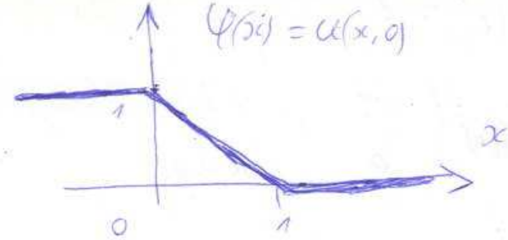


$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = \varphi(x) = \begin{cases} 1, & x \le 0 \\ 1-x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases} \end{cases}$$

$$\begin{cases} \dfrac{dx(t)}{dt} = u(x(t),t) \\ \dfrac{du(x(t),t)}{dt} = 0 \end{cases} \Longrightarrow \begin{cases} \dfrac{dx}{dt} = u(x(0),0) \\ u(x(t),t) = u(\underbrace{x(0)}_{=x_0},0) = \varphi(x_0) \end{cases} \Rightarrow$$

$\Rightarrow \quad x = x_0 + t \cdot \varphi(x_0)$

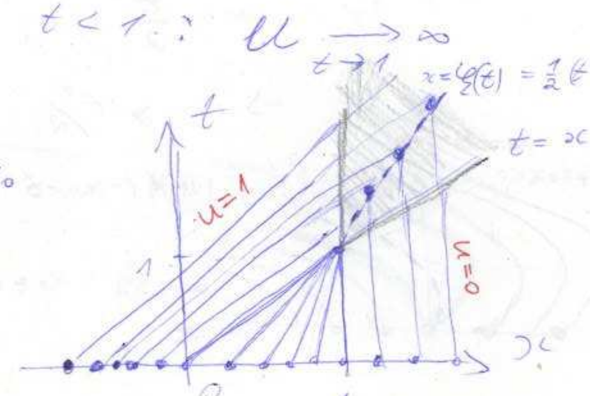
1) $x_0 < 0$: $\varphi(x_0) = 1 \Rightarrow x = x_0 + t \Rightarrow x_0 = x - t$

$u(x,t) = \varphi(x_0) = 1$; $x_0 < 0 \Rightarrow x < t$

2) $0 \le x_0 \le 1$: $\varphi(x_0) = 1 - x_0 \Rightarrow x = x_0 + (1-x_0)t$

$\Rightarrow x_0 = \dfrac{x-t}{1-t} \Rightarrow u(x,t) = \varphi(x_0) = \dfrac{1-x}{1-t}$

Solution is valid only while $t < 1$: $u \xrightarrow[t \to 1]{} \infty$



3) $x_0 > 1$: $\varphi(x_0) = 0 \Rightarrow x = x_0$

$u(x,t) = \varphi(x_0) = 0$

After $t = 1$ the shock is formed.

Rankine-Hugoniot condition tells us the speed of shock wave and at the same time gives us the line $x = \xi(t)$ in (xt)-plane (somewhere in the shaded region) across which the discontinuity of solution occurs:

$$\frac{\Big[\overbrace{u(\xi(t)+0,t)}^{=0,\ \text{right limit}}\Big]^2\!/2 - \Big[\overbrace{u(\xi(t)-0,t)}^{=1,\ \text{left limit}}\Big]^2\!/2}{\underbrace{u(\xi(t)+0,t)}_{=0} - \underbrace{u(\xi(t)-0,t)}_{=1}} = \frac{0 - 1/2}{0 - 1} = 1/2 = \frac{dx}{dt} = \xi'(t) \equiv s$$

$\varphi'(t) = \frac{1}{2} \Rightarrow \varphi(t) = \frac{1}{2}t + C_0$

Since shock line passes through $(x,t) = (1,1)$ point, we must have: $\varphi(1) = 1 = 1/2 + C_0 \Rightarrow C_0 = 1/2$

$x = \varphi(t) = \frac{1}{2}(t+1)$ – in this case the shock occurs across the straight line on (xt)-plane.

Let's verify entropy condition: $\underbrace{u(\varphi(t)-0,\, t)}_{=1} > \underset{\varphi'(t) = \frac{1}{2}}{s} > \underbrace{u(\varphi(t)+0,\, t)}_{=0} \Rightarrow 1 > \frac{1}{2} > 0$ ✓

Finally, the solution is (as we read from (xt)-plane):

For $t < 1$:

$$u(x,t) = \begin{cases} 1, & x < t \\ \frac{1-x}{1-t}, & t \le x \le 1 \\ 0, & x > 1 \end{cases}$$

formation of shock wave (steepening of wave front)

For $t > 1$:

$$u(x,t) = \begin{cases} 1, & x < \frac{1}{2}(t+1) \\ 0, & x > \frac{1}{2}(t+1) \end{cases}$$

propagation of shock wave with velocity 1/2