Tutorial # 5

N 3.2.1

$$\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < +\infty \\ u_x(0,t) = 0, & t > 0 \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), & 0 < x < +\infty \end{cases}$$

└ Neumann problem

In order to apply D'Alembert formula only valid for $-\infty < x < \infty$, we want to extend initial data $\varphi(x), \psi(x)$ defined for $x > 0$ in such a way that the original boundary condition will not be affected (i.e. will be automatically satisfied).

Key observation: $u_x(0,t) = 0$ if $\varphi(x), \psi(x)$ are extended as even functions ✓ (φ_0, ψ_0 defined on all axis): $\varphi_0(-x) = \varphi_0(x)$, $\psi_0(-x) = \psi_0(x)$.

Indeed:

$$u(x,t) = \frac{1}{2}\left[\varphi_0(x+ct) + \varphi_0(x-ct)\right] + \frac{1}{2c} \overbrace{\int_{x-ct}^{x+ct} \psi_0(\xi)\,d\xi}^{= \int_{x-ct}^{0} \psi(\xi)d\xi + \int_0^{x+ct} \psi(\xi)d\xi}$$

$$u_x(x,t) = \frac{1}{2}\left[\varphi_0'(x+ct) + \varphi_0'(x-ct)\right] + \frac{1}{2c}\left[-(-c)\psi_0(x-ct) + c\cdot\psi_0(x+ct)\right] =$$

$$= \frac{1}{2}\left[\varphi_0'(x+ct) + \varphi_0'(x-ct)\right] + \frac{1}{2}\left[\psi_0(x-ct) + \psi_0(x+ct)\right]$$

$$u_x(0,t) = \underbrace{\frac{1}{2}\left[\varphi_0'(ct) + \overset{= -\varphi_0'(ct)}{\varphi_0'(-ct)}\right]}_{=0} + \underbrace{\frac{1}{2}\left[\underset{= -\psi_0(ct)}{\psi_0(-ct)} + \psi_0(ct)\right]}_{=0} = 0$$

since $\varphi(-x) = \varphi(x)$ implies $-\varphi'(-x) = \varphi'(x)$ (by direct differentiation)

$$u(x,t) = \begin{cases} \frac{1}{2}\left[\varphi(x+ct) + \overset{= \varphi(-x+ct)}{\varphi_0(x-ct)}\right] + \frac{1}{2c}\left[\overset{= \int_0^{-x+ct} \psi(\xi)d\xi}{\int_{x-ct}^{0} \psi_0(\xi)d\xi} + \int_0^{x+ct} \psi(\xi)d\xi\right], & 0 < x < ct \\ \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)d\xi, & x > ct \end{cases}$$

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$= c^2 \Rightarrow c = 2$

$$\begin{cases} u_{tt} = 4u_{xx}, & 0 < x < +\infty \\ u(0,t) = 0, & t > 0 \\ \underbrace{u(x,0) = 1}_{=\varphi(x)}, \; \underbrace{u_t(x,0) = 0}_{=\psi(x)}, & 0 < x < +\infty \end{cases}$$

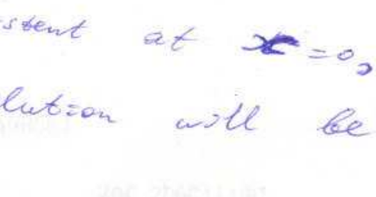
non-consistent at $x=0, t=0$: $u(0,0) = 0 \neq 1$

$\Rightarrow$ solution will be discontinuous.

Want to apply general reflection method (for Dirichlet problem), therefore we need to extend initial data in odd way:

$$\varphi_0(-x) = -\varphi(x), \quad \underbrace{\psi_0(-x) = -\psi(x)}_{=0 \qquad =0}, \quad -\infty < x < +\infty$$

extension yields a discontinuity: — nothing to extend, works automatically



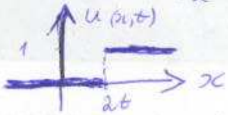
$$\varphi_0(z) = \begin{cases} 1, & z > 0 \\ -1, & z < 0 \end{cases}$$

D'Alembert formula: for $x, t > 0$

$$u(x,t) = \frac{1}{2}\Big[\underbrace{\varphi_0(x+2t)}_{=1} + \varphi_0(x-2t)\Big] \overset{\equiv 0}{} = , \quad x \geq 0, \; t \geq 0$$

$$= \begin{cases} 1, & x > 2t \\ 0, & x < 2t \end{cases}$$

— note that this is not diff-ble, however it still satisfies the wave eq. in "weak" sense (recall shock waves)



$\Rightarrow$ the discontinuity (singularity) propagates as $x = 2t$