(№ 4.1.2)

Tutorial # 6

$$\begin{cases} u_t = k u_{xx}, & 0 < x < l,\ t > 0 \\ u(x,0) = 1, & 0 \le x \le l \\ u(0,t) = u(l,t) = 0, & t > 0 \end{cases}$$

$u = 1$ at $t = 0$

0 l x

$u = 0$ for $t > 0$

Use separation of variables technique

$$u(x,t) = X(x) \cdot T(t)$$

$$u_t(x,t) = k u_{xx}(x,t) \Rightarrow X T' = k X'' T \Rightarrow$$

$: (T \cdot X \cdot k)$

function of t only, function of x only

$$\Rightarrow \frac{T'(t)}{k T(t)} = \frac{X''(x)}{X(x)} = -\lambda = \text{const}$$

(2) (1)

$$(1):\quad X'' + \lambda X = 0$$

$$X(x) = C_1 \cdot \sin(\sqrt{\lambda}\, x) + C_2 \cdot \cos(\sqrt{\lambda}\, x)$$

$$u(0,t) = 0 \Rightarrow X(0) = 0 : \quad 0 + C_2 \cdot 1 = 0 \Rightarrow C_2 = 0$$

$$u(l,t) = 0 \Rightarrow X(l) = 0 : \quad C_1 \cdot \sin(\sqrt{\lambda}\, l) = 0 \Rightarrow \sqrt{\lambda}\, l = \pi m,\ m \in \mathbb{Z}$$

$\ne 0$ – otherwise $u(x,t) = 0$, but $u(x,0) = 1$ violated

$\ne \{0, \dots\}$

$$\Rightarrow \lambda = \frac{\pi^2 m^2}{l^2}$$

$$X(x) = C \cdot \sin\left(\frac{\pi m}{l} x\right) \quad \text{for } \forall m \in \mathbb{Z}^+$$

$$(2):\quad T'(t) + \lambda k \cdot T(t) = 0 \iff \frac{T'(t)}{T(t)} = -\lambda k$$

$$T(t) = A \cdot e^{-\lambda k t}$$

To All MQF Applicants:

In the interest of assisting you in your entrance interview for the Master of Quantitative Finance program I am sending along some information and a few suggestions.

Your interview will take place with two interviewers, who teach or supervise in the MQF program. There will be a range of questions including technical questions similar to those on the entrance test, some analytical or brain teaser questions and some career related questions.

Suggestions for all interviews

1. Be prepared to give a one or two minute commentary or summary about yourself and why you are a good candidate for this program.
2. If you do not understand a question ask the interviewers to repeat or re-word the question.
3. If you have no idea about the topic, tell us you are not familiar with the concept. That way we can move on to another question that you can answer.
4. For brain teaser questions or analytical questions, think about your options and walk us through the process which you might follow to determine an answer. Your approach to the question is often more important than the final answer.
5. Towards the end of your interview you will have the opportunity to ask the faculty about the program so prepare your questions in advance.

For in person interviews:

- Recommended dress is business casual or business
- Be sure to arrive in advance, particularly if you are not familiar with the campus or the specific building
- A friendly, firm handshake is always a good start and a smile is welcome!
- Bring 2 copies of your resume

For telephone interviews:

- Provide us with the telephone number to a regular (land) line. We will call you and pay for the charges
- Ensure that you have complete privacy for the interview (no room mates or family present)
- Do have a pen and some paper for scribbling or figuring out an answer
- Do have a glass of water – in case your throat gets dry
- Do have your resume in front of you

Best of luck with your interview!

Regards
Mary Rae, Program Coordinator MQF

Hence

$$U(x,t) = X(x)\cdot T(t) \overset{A\cdot C}{=} B \cdot e^{-\frac{\pi^2 m^2}{\ell^2} k t} \cdot \sin\left(\frac{\pi m}{\ell} x\right), \quad m \in \mathbb{Z}^+$$

this satisfies the PDE and boundary condition, so is any linear combination of this solutions for $\forall m$:

$$U(x,t) = \sum_{m=0}^{\infty} B_m \cdot e^{-\frac{\pi^2 m^2}{\ell^2} k t} \cdot \sin\left(\frac{\pi m}{\ell} x\right) \quad \text{- general solution}$$

We want to choose constants B_m s.t. the initial condition is also satisfied:

$$U(x,0) = 1 = \sum_{m=0}^{\infty} B_m \cdot \sin\left(\frac{\pi m}{\ell} \cdot x\right)$$

As a hint we're given the following representation of the unity:

$$1 = \frac{4}{\pi} \cdot \sum_{m=0}^{\infty} \frac{1}{2m+1} \cdot \sin\left(\frac{\pi(2m+1)}{\ell} x\right) =$$

$$= \frac{4}{\pi}\left[\sin\left(\frac{\pi x}{\ell}\right) + \frac{1}{3}\cdot\sin\left(\frac{3\pi}{\ell}\cdot x\right) + \frac{1}{5}\sin\left(\frac{5\pi}{\ell}\cdot x\right) + \dots\right]$$

Comparing this with above, we conclude:

$$\left[\begin{array}{ll} B_m = 0 & \text{for } m\text{ - even} \\ B_m = \frac{1}{m}\cdot\frac{4}{\pi} & \text{for } m\text{ - odd} \end{array}\right.$$

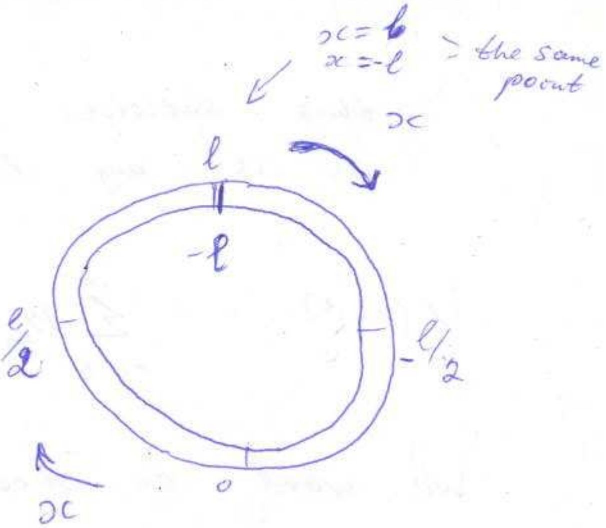
Finally:

$$U(x,t) = \frac{4}{\pi} \sum_{m - odd}^{\infty} \frac{1}{m} e^{-\frac{\pi^2 m^2}{\ell^2} k t} \sin\left(\frac{\pi m}{\ell}\cdot x\right) =$$

$$= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} e^{-\frac{\pi^2 (2n+1)^2}{\ell^2} k t} \cdot \sin\left(\frac{\pi(2n+1)}{\ell} x\right)$$

N 4.2.7

$$\begin{cases} u_t = k u_{xx}, & -l \le x \le l \\ u(-l,t) = u(l,t) \\ u_x(-l,t) = u_x(l,t) \end{cases}$$



As before:

$$u(x,t) = X(x) \cdot T(t)$$

$$\Rightarrow \quad \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(l) = X(-l) \\ X'(l) = X'(-l) \end{cases} \Rightarrow X(x) = A \cdot \cos(\sqrt{\lambda} x) + B \cdot \sin(\sqrt{\lambda} x)$$

$$X'(x) = -A\sqrt{\lambda} \sin(\sqrt{\lambda} x) + B\sqrt{\lambda} \cos(\sqrt{\lambda} x)$$

$$A \cdot \cos(\sqrt{\lambda} l) + B \cdot \sin(\sqrt{\lambda} l) = A\cos(\sqrt{\lambda} l) - B\sin\sqrt{\lambda} l$$

$$\Downarrow \quad 2B \cdot \sin(\sqrt{\lambda} l) = 0$$

$$-A\sqrt{\lambda} \cdot \sin(\sqrt{\lambda} l) + B\sqrt{\lambda} \cos(\sqrt{\lambda} l) = A\sqrt{\lambda} \sin(\sqrt{\lambda} l) + B\sqrt{\lambda} \cos\sqrt{\lambda} l$$

$$\Downarrow \quad 2A\sqrt{\lambda} \cdot \sin(\sqrt{\lambda} l) = 0$$

Hence

$$\begin{cases} 2B \cdot \sin(\sqrt{\lambda} l) = 0 \\ 2A\sqrt{\lambda} \cdot \sin(\sqrt{\lambda} l) = 0 \end{cases} \Rightarrow \sin(\sqrt{\lambda} l) = 0$$

$$\Downarrow \sqrt{\lambda} l = \pi n, \quad n \in \mathbb{Z}^+ = \{0, 1, 2, \dots\}$$

$$\Downarrow \lambda = \frac{\pi^2 n^2}{l^2}$$

allowing both
$A, B \neq 0$ to be
able to solve the same
problem for arbitrary initial condition

$$\frac{T'(t)}{k\cdot T(t)} = -\lambda \iff T' + \lambda k\cdot T = 0$$

$$T(t) = C\cdot e^{-\lambda k t} = C\cdot e^{-\frac{\pi^2 n^2}{\ell^2} k t}$$

$$u(x,t) = X(x)\cdot T(t) = \sum_{n=0}^{\infty}\left[A_n\cdot\cos\left(\frac{n\pi x}{\ell}\right) + B_n\sin\left(\frac{n\pi x}{\ell}\right)\right] e^{-\frac{\pi^2 n^2}{\ell^2} k t} =$$

$$= A_0 + \sum_{n=1}^{\infty}\left[A_n\cdot\cos\left(\frac{n\pi x}{\ell}\right) + B_n\cdot\sin\left(\frac{n\pi x}{\ell}\right)\right] e^{-\frac{\pi^2 n^2}{\ell^2} k t}$$

arbitrary $\Rightarrow$ can rename into $\frac{1}{2}A_0$ for future convenience