Tutorial #7

N5.1.5

Recall: $\varphi(x) = \sum_{n=1}^{\infty} A_n \cdot \sin\frac{n\pi x}{l}$, $\psi(x) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n \cos\frac{n\pi x}{l}$, $x \in (0, l)$

$A_n = \frac{2}{l}\int_0^l \varphi(x)\cdot \sin\frac{n\pi x}{l}\,dx$, $B_n = \frac{2}{l}\int_0^l \psi(x)\cdot\cos\frac{n\pi x}{l}\,dx$, $n \in \mathbb{N}$

$$\varphi(x) = x = \sum_{n=1}^{\infty} A_n \cdot \sin\frac{n\pi x}{l}, \quad 0 < x < l$$

$$A_n = \frac{2}{l}\cdot\int_0^l x \cdot \sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\left[-\frac{l}{n\pi}x\cdot\cos\frac{n\pi x}{l}\Big|_0^l + \right.$$

$$\left. + \frac{l}{n\pi}\int_0^l \cos\frac{n\pi x}{l}\,dx\right] = -\frac{2l}{n\pi}\cdot\underbrace{\cos(n\pi)}_{=(-1)^n} = (-1)^{n+1}\cdot\frac{2l}{n\pi}$$

$$\int \left| x = \frac{2l}{\pi}\cdot\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\sin\frac{n\pi x}{l}\right| \cdot dx$$

$$\frac{x^2}{2} + C = \frac{2l}{\pi}\cdot\left(-\frac{l}{\pi}\right)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{l}$$

$$\frac{x^2}{2} = -C + 2\left(\frac{l}{\pi}\right)^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{l}$$

On the other hand:

$$\frac{x^2}{2} = \underbrace{\frac{B_0}{2}}_{=-C} + \sum_{n=1}^{\infty}\underbrace{B_n}_{=2\left(\frac{l}{\pi}\right)^2\cdot\frac{(-1)^n}{n^2}}\cdot\cos\frac{n\pi x}{l}$$

$$B_0 = \frac{2}{l}\cdot\int_0^l \frac{x^2}{2}\cdot\underbrace{\cos\left(\frac{0\cdot\pi x}{l}\right)}_{=1}dx = \frac{l^2}{3}$$

$$\Rightarrow \frac{x^2}{2} = \frac{l^2}{6} + 2\left(\frac{l}{\pi}\right)^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cdot\cos\frac{n\pi x}{l}$$

$x = 0$: $\quad 0 = +l^2/6 + 2\,l^2/\pi^2\cdot\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$

$x = l: \quad l^2/2 = +l^2/6 + 2\, l^2/\pi^2 \cdot \sum_{n=1}^{\infty} \frac{\overbrace{(-1)^n \cdot (-1)^n}^{1}}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6}\pi^2$

№ 5.2.4

Full Fourier series:

$$\varphi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left(A_n \cdot \cos \frac{n\pi x}{l} + B_n \cdot \sin \frac{n\pi x}{l} \right), \quad x \in (-l, l)$$

$$A_n = \frac{1}{l} \cdot \int_{-l}^{l} \varphi(x) \cdot \cos \frac{n\pi x}{l} \, dx$$

$$B_n = \frac{1}{l} \cdot \int_{-l}^{l} \varphi(x) \cdot \sin \frac{n\pi x}{l} \, dx$$

Let $\varphi(x)$ be odd: $\varphi(x) = -\varphi(-x)$

$$\varphi(x) + \varphi(-x) = 0 = 2 \cdot \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left[A_n \left(\cos \frac{n\pi x}{l} + \cos\left(-\frac{n\pi x}{l}\right) \right) + B_n \left(\sin \frac{n\pi x}{l} + \sin\left(-\frac{n\pi x}{l}\right) \right) \right] = A_0 + 2 \sum_{n=1}^{\infty} A_n \cdot \cos \frac{n\pi x}{l}$$

But $\left\{1, \cos \frac{n\pi x}{l}\right\}_{n \in \mathbb{Z}^+ := \{1,2,3,\dots\}}$ are linearly independent.

Therefore:

$$A_0 + 2 \sum_{n=1}^{\infty} A_n \cdot \cos \frac{n\pi x}{l} = 0 \iff \begin{cases} A_0 = 0 \\ A_n = 0 \end{cases}, \quad n = 1, 2, \dots$$

Then:

$$\varphi(x) = \sum_{n=1}^{\infty} B_n \cdot \sin \frac{n\pi x}{l}$$

Now consider even $\varphi(x)$: $\varphi(x) = \varphi(-x)$

Certainly, we can argue as before, but instead let's compute:

$$B_n = \frac{1}{l} \cdot \int_{-l}^{l} \varphi(x) \cdot \sin\frac{n\pi x}{l}\, dx = \frac{1}{l}\Big[\int_0^l \varphi(x) \sin\frac{n\pi x}{l}\, dx +$$

$$+ \underbrace{\int_{-l}^{0} \varphi(x) \sin\frac{n\pi x}{l}\, dx}_{= \int_0^l \varphi(-y)\cdot \sin\left(-\frac{n\pi y}{l}\right) dy = -\int_0^l \varphi(-x)\cdot \sin\frac{n\pi x}{l}\, dx} \Big] = \frac{1}{l}\int_0^l \big[\underbrace{\varphi(x) - \overset{\varphi(x)}{\overset{\parallel}{\varphi(-x)}}}_{=0}\big] \sin\frac{n\pi x}{l}\, dx = 0$$

i.e. $B_n = 0$, $n = 1, 2, \ldots$

Then:

$$\varphi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cdot \cos\frac{n\pi x}{l}$$