Tutorial #8

№ 5.3.4 (a, b)

$$\begin{cases} u_t = k u_{xx}, & 0 < x < l \\ u(0,t) = U \\ u_x(l,t) = 0 \\ u(x,0) = 0 \end{cases} \xrightarrow{v(x,t) = u(x,t) - U} \begin{cases} v_t = k v_{xx} \\ v(0,t) = 0 \\ v_x(l,t) = 0 \\ v(x,0) = -U \end{cases}$$

$$v(x,t) = X(x) \cdot T(t) \quad \Rightarrow \quad \frac{T'}{kT} = \frac{X''}{X} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0, \; X'(l) = 0 \end{cases} \Rightarrow \quad X(x) = A \cdot \sin(\sqrt{\lambda}\, x) + B \cdot \cos(\sqrt{\lambda}\, x)$$

$$\Rightarrow B = 0 \quad \Rightarrow \cos(\sqrt{\lambda}\, l) = 0 \quad \Rightarrow \sqrt{\lambda}\, l = \frac{\pi}{2} + \pi n$$

$$\Rightarrow \lambda = \frac{\pi^2}{l^2}\left(n + \tfrac{1}{2}\right)^2$$

$$v(x,t) = \sum_{n=0}^{\infty} A_n \cdot e^{-\frac{\pi^2 (2n+1)^2}{4l^2} k t} \cdot \sin\left(\frac{\pi(2n+1)}{2l} x\right)$$

$$v(x,0) = -U = \sum_{n=0}^{\infty} A_n \sin\left[\frac{\pi(2n+1)}{2l} x\right]$$

$$A_n = -\frac{2U}{l} \int_0^l \sin\left(\frac{\pi(2n+1)}{2l} x\right) dx = \frac{2U \cdot 2l}{\cancel{l}\, \pi(2n+1)} \cos\left(\frac{\pi(2n+1)x}{2l}\right)\Bigg|_0^l =$$

$$= -\frac{4U}{\pi(2n+1)}$$

$$v(x,t) = \sum_{n=0}^{\infty} -\frac{4U}{\pi(2n+1)}\, e^{-\frac{\pi^2(2n+1)^2}{4l^2}\kappa t} \cdot \sin\left(\frac{\pi(2n+1)}{2l}x\right)$$

Does this series converge?

Consider the series $\sum_{n=0}^{\infty} \underbrace{\frac{1}{2n+1}\, e^{-\frac{\pi^2(2n+1)^2}{4l^2}\kappa t} \cdot \sin\left(\frac{\pi(2n+1)}{2l}x\right)}_{=:\, v_n(x,t)}$

It converges if $\sum_{n=0}^{\infty} |v_n(x,t)| < \infty$

But $|v_n(x,t)| \le \underbrace{1}_{\ge \frac{1}{2n+1}} \cdot e^{-\frac{\pi^2(2n+1)^2\kappa t}{4l^2}} \cdot \underbrace{1}_{\ge |\sin(\dots)|}$ — independent of $x \Rightarrow$ convergence will be uniform

$\sum_{n=0}^{\infty} |v_n(x,t)|$ will converge by comparison test

if $\sum_{n=0}^{\infty} e^{-\overbrace{\frac{\pi^2\kappa t}{4l^2}}^{=:a}\cdot(2n+1)^2}$ converges, and it does

Since $\sum_{n=0}^{\infty} e^{-a(2n+1)^2} < \sum_{m=0}^{\infty} e^{-am^2}$

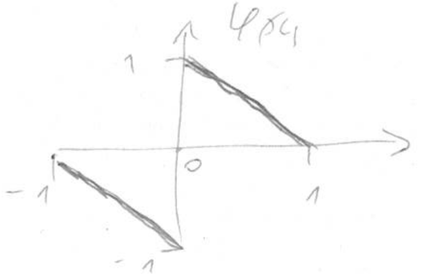
The last series converges by integral test:

$$\int_1^{\infty} e^{-ax^2}dx < \int_0^{\infty} e^{-ax^2}dx = \frac{1}{2}\sqrt{\pi/a} < \infty$$ for $a>0$ (i.e. $t>0$)

Hence $v(x,t)$ series converges uniformly for $t>0$, so the solution to the original problem is defined and given by

$$u(x,t) = v(x,t) + U = U\cdot\Big[1 - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\, e^{-\frac{\pi^2(2n+1)^2}{4l^2}\kappa t} \cdot \sin\left(\frac{\pi(2n+1)}{2l}x\right)$$

N 5.4.7



$$\varphi(x) = \begin{cases} -1 - x, & -1 < x < 0 \\ 1 - x, & 0 < x < 1 \end{cases}$$

What is full Fourier series of $\varphi(x)$?

Formally: $\varphi(x) = \sum\limits_{n=-\infty}^{\infty} C_n \cdot e^{\frac{i n \pi x}{l}}$, but we anticipate expansion over only odd functions (i.e. sines)

$$C_n = \frac{1}{2l} \cdot \int_{-l}^{l} \varphi(x) \cdot e^{-\frac{i\pi n x}{l}} dx$$

In our case $l = 1$

Since $\varphi(x)$ is odd, $C_0 = 0$

For $n \neq 0$:

$$C_n = \frac{1}{2}\left[-\underbrace{\int_{-1}^{0}(1+x)\, e^{-i\pi n x} dx}_{= \frac{i(1+x)}{\pi n} e^{-i\pi n x}\Big|_{-1}^{0} - \frac{i}{\pi n}\underbrace{\int_{-1}^{0} e^{-i\pi n x} dx}_{= 1 - (-1)^n}} + \underbrace{\int_{0}^{1}(1-x)\, e^{-i\pi n x} dx}_{= \dots} \right] =$$

$$= -\frac{i}{\pi n} - \frac{1}{2\pi^2 n^2}\underbrace{\left(-\left[1 - (-1)^n\right] - \left[(-1)^n - 1\right] \right)}_{= 0} = -\frac{i}{\pi n}$$

$$\varphi(x) = -\sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \frac{i}{\pi n} e^{i\pi n x} = +\sum_{n=1}^{+\infty} \frac{i}{\pi n} e^{i\pi n x} + \underbrace{\sum_{n=-1}^{-\infty} \frac{i}{\pi n} e^{i\pi n x}}_{= -\sum\limits_{n=1}^{+\infty} \frac{i}{\pi n} e^{-i\pi n x}} =$$

$$= -\frac{i}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\underbrace{\left(e^{i\pi n x} - e^{-i\pi n x} \right)}_{= 2i \cdot \sin(n\pi x)} = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin(n\pi x)$$

$\varphi(x) \in L^2(-1,1): \int_{-1}^{1} |\varphi(x)|^2 dx < \infty$

by Th. 3 of L^2-convergence

$\Rightarrow$ the series converges in mean-square sense

Now notice $\varphi(x)$, $\varphi'(x)$ are piecewise-continuous

by Th. 4 of pointwise convergence

$\Rightarrow$ the series converges pointwise

for $\forall x \in (-1, 0) \cup (0, 1)$

But uniform convergence breaks down

due to discontinuity at $x = 0$:

$$\underbrace{\lim_{N\to\infty} \max_{x\in(-1,1)} \Big| \underset{\substack{\to \pm 1 \\ \text{as } x \to 0\pm}}{\varphi(x)} - \underbrace{\frac{2}{\pi} \sum_{n=1}^{N} \frac{1}{n} \sin(n\pi x)}_{\to 0 \text{ as } x \to 0} \Big|}_{=1} = 1 \neq 0$$