Tutorial # 9

Sample midterm Problem #2

a) $$\begin{cases} X'' = -\lambda X \\ X'(0) = 0 \\ X(1) + \alpha \cdot X'(1) = 0 \end{cases}$$

$$\int_0^1 X \cdot \left| X'' = -\lambda X \right| \cdot dx \quad \Rightarrow$$

$$\Rightarrow \underbrace{\int_0^1 X \cdot X'' dx}_{= \underbrace{X \cdot X' \Big|_0^1}_{= -\alpha (X'(1))^2} - \int_0^1 (X')^2 dx} = -\lambda \int_0^1 X^2 dx$$

due to boundary conditions

$$\Rightarrow \underbrace{\alpha \cdot (X'(1))^2}_{> 0} + \underbrace{\int_0^1 (X')^2 dx}_{> 0} = \lambda \underbrace{\int_0^1 X^2 dx}_{> 0}$$

Hence, $\alpha \geq 0$ implies $\lambda > 0$, i.e. all eigenvalues are positive в) $\begin{cases} u_t + 4t\cdot u = u_{xx} \\ u(0,t) = u(1,t) = 0 \end{cases}$

$$u(x,t) = X(x)\cdot T(t) \quad \Rightarrow \quad \frac{T'}{T} + 4t = \frac{X''}{X} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 = X(1) \end{cases} \Rightarrow \quad X_n(x) = \sin(\underbrace{\pi n}_{=\sqrt{\lambda}} x) \Rightarrow \lambda_n = \pi^2 n^2$$

$$T' + 4t\cdot T = -\lambda T$$

$$T' + (4t+\lambda)T = 0$$

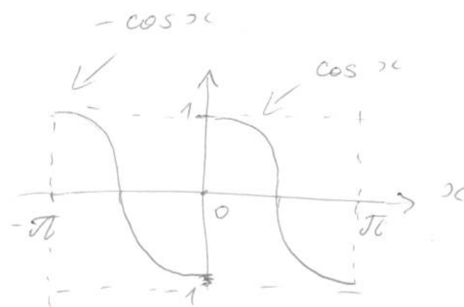
$$\left(e^{2t^2+\lambda t}\cdot T\right)' = 0$$

$$T(t) = c\cdot e^{-2t^2-\lambda t}$$

$$u(x,t) = e^{-2t^2}\cdot\sum_{n=1}^{\infty} c_n\cdot e^{-\lambda_n t}\cdot\sin(\pi n x)$$

$$= e^{-2t^2}\sum_{n=1}^{\infty} c_n\cdot e^{-\pi^2 n^2 t}\cdot\sin(\pi n x)$$

N 5.4.6



$$f(x) = \begin{cases} \cos x, & 0 < x < \pi \\ -\cos x, & -\pi < x < 0 \end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} a_n \cdot \sin(nx)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\overset{\text{odd}}{f(x)} \cdot \overset{\text{odd}}{\sin(nx)}}_{\text{even}} \, dx = \frac{2}{\pi} \int_0^{\pi} \cos x \cdot \sin(nx) =$$

$$= \frac{1}{\pi} \left\{ \int_0^{\pi} \sin[(n+1)x] \, dx + \int_0^{\pi} \sin[(n-1)x] \, dx \right\} \overset{\substack{\text{assume} \\ n \neq 1}}{=}$$

$$= -\frac{1}{\pi} \left[\frac{\overbrace{\cos(\pi(n+1))}^{=-(-1)^n = (-1)^{n+1}} - 1}{n+1} + \frac{\overbrace{\cos(\pi(n-1))}^{=-(-1)^n = (-1)^{n-1}} - 1}{n-1} \right] =$$

$$= \frac{1}{\pi} \frac{[(-1)^n + 1] \cdot (n - \cancel{1} + n + \cancel{1})}{n^2 - 1} = \frac{2}{\pi} \cdot \frac{n}{n^2-1} \cdot [(-1)^n + 1], \quad n \neq 1$$

$n = 1$:

$$a_1 = \frac{1}{\pi} \cdot \int_0^{\pi} \sin(2x) \, dx = -\frac{1}{2\pi} \cos(2x) \Big|_0^{\pi} = 0$$

$$f(x) = \frac{2}{\pi} \cdot \sum_{n=2}^{\infty} \frac{n}{n^2-1} [(-1)^n + 1] \cdot \sin(nx)$$

$$f'(x) = \frac{4}{\pi} \sum_{\substack{n=2 \\ n\text{-even}}}^{\infty} \frac{n}{n^2-1} \sin(nx) = \begin{cases} \cos x, & 0 < x < \pi \\ -\cos x, & -\pi < x < 0 \\ 0, & x = \pm\pi, 0 \end{cases}$$

by pointwise convergence
($f(x)$, $f'(x)$ are piecewise continuous)