Chapter 1

Asymptotic solutions of convolution integral equations with even positive definite kernels on small or large intervals

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Abstract. *One-dimensional convolution integral equations with real-valued regular symmetric kernels naturally arise in a number of inverse problems for elliptic PDEs when field values on an one-dimensional set are known whereas the source term is not. In certain geometrical configurations this can be rephrased as an issue of inversion of a compact self-adjoint integral operator, a problem that can be solved by constructing an approximation of the resolvent of this operator, given its approximate eigendecomposition. Motivated by this application and number of other contexts where convolution integral operators play a crucial role, we consider a generic eigenvalue problem for a self-adjoint convolution integral operator on an interval where the kernel is real-valued even C^1-smooth function which (in case of large interval) is absolutely integrable on the real line. We show how this spectral problem can be solved by two different asymptotic techniques that take advantage of the size of the interval. In case of small interval, this is done by approximation with an integral operator for which there exists a commuting differential operator thereby reducing the problem to a boundary-value problem for second-order ODE, and often giving the solution in terms of explicitly available special functions such as prolate spheroidal harmonics. In case of large interval, the solution hinges on solvability, by Riemann-Hilbert approach, of an approximate auxiliary integro-differential half-line equation of Wiener-Hopf type, and culminates in simple characteristic equations for eigenvalues, and, with such an approximation to eigenvalues, approximate eigenfunctions are given in an explicit form. We note that, unlike finite-rank approximation of a compact operator, the auxiliary problems arising in both small- and*

large-interval cases admit infinitely many solutions (eigenfunctions) and hence structurally better represent the original integral operator.

We are concerned with a convolution Fredholm equation of the second kind on a finite interval. That is, given constants $a > 0$, $\lambda \in \mathbb{C}$ and functions $K(x)$, $g(x)$, the problem consists in finding functions $f(x)$ which satisfy

$$\int_A K(x-t) f(t)\, dt = \lambda f(x) + g(x), \quad x \in A := (-a, a). \tag{1.1}$$

Equations of this type arise in many applications ranging from a variety of physical problems to the theory of signal processing [2, 7], stochastic processes [5], analytic function approximation [18] and construction of weighted orthogonal bases [21]. Among particular applications, to name a few, are the problems of geological prospecting [15], electrostatics and hydrodynamics problems with circular geometry [13, 14], neutron transport and radiative transfer in a slab [19, 30], study of thermodynamic limits Bose or Fermi quantum gases [28, 29] and antiferromagnetic Heisenberg chain [9].

In particular, we will be dealing with a homogeneous version of (1.1), a problem tantamount (under appropriate restriction on a class of kernel functions) to finding a spectral decomposition of a convolution integral operator defined by the kernel function $K(x)$. In other words, we are interested in finding the set of all pairs $(\lambda, f) \in \left(\mathbb{R}, L^2(A)\right)$ such that

$$\int_A K(x-t) f(t)\, dt = \lambda f(x), \quad x \in A. \tag{1.2}$$

We note that equation (1.2) is a more delicate version of (1.1) as it requires more sophisticated analysis. On the other hand, knowledge of complete set of solutions of (1.2) allows solving (1.1), but also, as it is often the case, the solution procedure of (1.2) can be modified (simplified) to deal directly with (1.1). The advantage of solving (1.2) over (1.1) is also that the spectral decomposition permits solving much more general operator problems such as the one arising in approximation theory with quadratic truncated convolution operator equation [18].

Equations (1.1)-(1.2) have been a subject of study for many years. We do not aim to review these works here, instead we just mention that while a closed form solution is not available except for very few special cases (such as polynomial or trigonometric kernel function), majority of efforts in this topic were focussed on study of bounds and asymptotical behavior of eigenvalues [4, 5, 31] whereas in direction of obtaining eigenfunctions usually one of two strategies is pursued.

One possibility is a reduction of the integral equation to an auxiliary problem which is also not explicitly solvable but can give some analytical insight or practical advantage (e.g. better convergence of an iterative numerical scheme, or construction of solutions for general right-hand sides $g(x)$ in terms of the auxiliary problem solution) [19, 26].

In the alternative strategy, one hopes to solve the original equation asymptotically in the case when the interval A is large. For constructive asymptotic solution techniques, apart from a crude periodic approximation [8], the typical assumptions on the kernel function are real-valuedness, even parity, positive definiteness, L^1-regularity and, also, the kernel function was supposed explicitly or implicitly (by imposing conditions on derivative of its Fourier transform at the origin) to have rapid (in most cases even exponential) decay at infinity [12, 16, 17, 23]. This last assumption is too restrictive as it does not allow (except some exceptional cases of (1.1) with particular values of λ [1, 13, 14, 28, 29]) one to treat even very simple (and yet important for applications listed above) kernels such as, for example, $\frac{1}{1+x^2}$ and $\frac{1}{(1+x^2)^{3/2}}$. It is the main purpose of this work to present a new method that allows dealing with kernels of algebraic decay. For the sake of simplicity of technical arguments, we restrict ourselves to the case when Fourier transform of the kernel function decays monotonically on the real line with distance from the origin, an assumption which is also often used [4, 12].

We also complement our method with a strategy to deal with equation (1.2) on small intervals. This part is meant to demonstrate an essentially different approach compared to commonly used finite-rank approximation of the integral operator obtained by series expansion of the kernel function.

We note that these results are generalization and significant simplification of the previous author's work related to Love/Lieb-Liniger/Gaudin equation [3] where the spectral problem for the integral operator with a particular kernel function $K(x) = \frac{1}{1+x^2}$ was treated.

1.1 Assumptions and some properties

Under assumption that $K \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$ is an even real-valued function, the convolution integral operator in the left-hand side of (1.2), that we denote $\mathfrak{K}$, is compact on $L^2(A)$ and self-adjoint. By the Spectral Theorem, spectrum of $\mathfrak{K}$ is purely discrete with real eigenvalues $(\lambda_n)_{n=1}^{\infty}$ converging to zero and corresponding eigenfunctions $(f_n)_{n=1}^{\infty}$ forming a basis of the range of $\mathfrak{K}$. Moreover, assuming that $\mathfrak{K}$ is positive definite (i.e. $\int_A F(x) \int_A K(x-t) F(t)\, dt dx > 0$ for any $F \in L^2(A)$) and that, for $k \in \mathbb{R}$, kernel's Fourier transform $\hat{K}(k) = \int_{\mathbb{R}} e^{2\pi i k x} K(x)\, dx$ decays strictly monotonically with increase of $|k|$, we have, from Parseval's identity, $0 < \lambda_n < \hat{K}(0)$, $n \in \mathbb{N}_+$. Positive definiteness can also be used to prove that eigenfunctions cannot vanish at the endpoints of the interval and the spectrum is simple (see [3, 7]). This, alongside with the parity of K, implies that each eigenfunction of $\mathfrak{K}$ is either odd or even and can be chosen to be real-valued. In what follows, we will also repeatedly use an elementary fact that Fourier transform of real-valued functions preserves their parity.

1.2 Solution for large A

Let us introduce, for $k \in \mathbb{R}$, $\kappa \in \mathbb{R}_+$,

$$\mathcal{G}(k,\kappa) := \frac{\hat{K}(\kappa)\left(k^2 - \kappa^2\right)}{\left(\hat{K}(\kappa) - \hat{K}(k)\right)(k^2+1)},$$

$$\mathcal{X}_+(k,\kappa) := \mathcal{G}^{1/2}(k,\kappa) \exp\left(\pm \frac{i}{2} \mathcal{H}\left[\log \mathcal{G}(\cdot,\kappa)\right](k)\right),$$

where $\mathcal{H}[F](k) := \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{F(\tau)}{k-\tau} d\tau$ is the Hilbert transform of F.

With help of these quantities, the main result can be formulated as follows.

Theorem 1 *Denote as* $\left(\kappa_n^{(e)}\right)_{n=1}^{\infty}$, $\left(\kappa_n^{(o)}\right)_{n=1}^{\infty}$ *all positive solutions of two sets of transcendental equations*

$$2\pi\kappa a - \arctan\frac{1}{\kappa} - \frac{1}{2}\mathcal{H}\left[\log \mathcal{G}(\cdot,\kappa)\right](\kappa) = \pi m, \quad m \in \mathbb{Z},$$

$$2\pi\kappa a - \arctan\frac{1}{\kappa} - \frac{1}{2}\mathcal{H}\left[\log\mathcal{G}\left(\cdot,\kappa\right)\right]\left(\kappa\right) = \pi\left(m+\frac{1}{2}\right), \quad m\in\mathbb{Z},$$

respectively, assuming that these sets are sorted in ascending order, i.e. $\kappa_{n-1}^{(e)} < \kappa_n^{(e)}$*,* $\kappa_{n-1}^{(o)} < \kappa_n^{(o)}$*,* $n\in\mathbb{N}_+$*. Eigenvalues corresponding to even and odd eigenfunctions then are given by*

$$\lambda_n^{(e)} = \hat{K}\left(\kappa_n^{(e)}\right) + \delta_{n,a}^{(e)}, \quad \lambda_n^{(o)} = \hat{K}\left(\kappa_n^{(o)}\right) + \delta_{n,a}^{(o)},$$

for some constants $\delta_{n,a}^{(e)}$*,* $\delta_{n,a}^{(o)}\in\mathbb{R}$ *such that* $\delta_{n,a}^{(o)}$*,* $\delta_{n,a}^{(e)}\to 0$ *as* $a\to+\infty$*, for any fixed* $n\in\mathbb{N}_+$*. Corresponding sets of even and odd eigenfunctions are, respectively, furnished by*

$$\begin{aligned} f_n^{(e)}(x) =& \frac{2\left(-1\right)^{n-1}}{\left[-2\kappa_n^{(e)}\left(\log\hat{K}\right)'\left(\kappa_n^{(e)}\right)\right]^{1/2}}\cos\left(2\pi\kappa_n^{(e)}x\right) \\ &+ \frac{i}{\pi}\int_{\mathbb{R}}\frac{e^{-2\pi ika}}{\left(k+i\right)\mathcal{X}_+\left(k,\kappa_n^{(e)}\right)}\left[\frac{\hat{K}\left(k\right)}{\hat{K}\left(k\right)-\hat{K}\left(\kappa_n^{(e)}\right)}\right. \\ &\left. - \frac{2\kappa_n^{(e)}}{\left(k^2-\left(\kappa_n^{(e)}\right)^2\right)\left(\log\hat{K}\right)'\left(\kappa_n^{(e)}\right)}\right]\cos\left(2\pi kx\right)dk + \mathcal{E}_{n,a}^{(e)}\left(x\right), \end{aligned}$$

$$\begin{aligned} f_n^{(o)}(x) =& \frac{2\left(-1\right)^{n-1}}{\left[-2\kappa_n^{(o)}\left(\log\hat{K}\right)'\left(\kappa_n^{(o)}\right)\right]^{1/2}}\sin\left(2\pi\kappa_n^{(o)}x\right) \\ &- \frac{1}{\pi}\int_{\mathbb{R}}\frac{e^{-2\pi ika}}{\left(k+i\right)\mathcal{X}_+\left(k,\kappa_n^{(o)}\right)}\left[\frac{\hat{K}\left(k\right)}{\hat{K}\left(k\right)-\hat{K}\left(\kappa_n^{(o)}\right)}\right. \\ &\left. - \frac{2\kappa_n^{(o)}}{\left(k^2-\left(\kappa_n^{(o)}\right)^2\right)\left(\log\hat{K}\right)'\left(\kappa_n^{(o)}\right)}\right]\sin\left(2\pi kx\right)dk + \mathcal{E}_{n,a}^{(o)}\left(x\right), \end{aligned}$$

with some $\left\|\mathcal{E}_{n,a}^{(e)}\right\|_{L^\infty(A)}$*,* $\left\|\mathcal{E}_{n,a}^{(o)}\right\|_{L^\infty(A)}\to 0$ *as* $a\to+\infty$*, for any fixed* $n\in\mathbb{N}_+$*.*

Proof: We give merely a sketch of the proof outlining main steps and skipping approximation error estimates.

Continuity and strict monotonic decay of $\hat{K}(k)$, along with the bounds $0<\lambda<\hat{K}(0)$, imply that there exists unique $k_0>0$ such that $\hat{K}(\pm k_0)=\lambda$.

Let us define, for $x\in\mathbb{R}\backslash A$, $f(x)=\frac{1}{\lambda}\int_A K(x-t)f(t)\,dt$.

It turns out that we can express solution of the original problem in terms of its own extension by continuity to the exterior of the interval A and, in particular, the half-line $x>a$. That is, we have

$$f(x) = \int_{\mathbb{R}\backslash A} R(x-t)f(t)\,dt = \int_a^\infty \left(R(x-t)\pm R(x+t)\right)f(t)\,dt, \quad x\in A, \qquad (1.3)$$

where plus and minus signs correspond to the cases of even and odd eigenfuntions, respectively, and

$$R(x) := \int_{\mathbb{R}} e^{-2\pi ikx} \left(\frac{\hat{K}(k)}{\hat{K}(k) - \lambda} - \frac{2\lambda}{\hat{K}'(k_0)} \frac{k_0}{k^2 - k_0^2} \right) dk + \frac{2\pi i\lambda}{\hat{K}'(k_0)} \exp\left(-2\pi i k_0 |x|\right), \quad x \in \mathbb{R}.$$

Therefore the problem is reduced to finding this continuous extension that we denote $\phi(x) := f(x+a)$, $x > 0$.

One can show that $\phi(x)$ satisfies the following approximate integro-differential equation

$$\phi''(x) + 4\pi^2 k_0^2 \phi(x) = \int_0^\infty \left(\frac{d^2}{dx^2} + 4\pi^2 k_0^2 \right) T(x-t)\,\phi(t)\,dt - \frac{8\pi^2 k_0 \lambda}{\hat{K}'(k_0)} \phi(x), \quad x > 0, \tag{1.4}$$

where

$$T(x) := \int_{\mathbb{R}} e^{-2\pi ikx} \left(\frac{\hat{K}(k)}{\hat{K}(k) - \lambda} - \frac{2\lambda}{\hat{K}'(k_0)} \frac{k_0}{k^2 - k_0^2} \right) dk, \quad x \in \mathbb{R}.$$

The standard Wiener-Hopf approach can be extended to treat equation (1.4). In particular, Fourier transform of its solution (extended by zero for $x < 0$) $\hat{\phi}(k) = \widehat{\chi_{\mathbb{R}_+}\phi}(k)\,dx$ is given by

$$\hat{\phi}(k) = \frac{i\phi(0)}{2\pi(k+i)\,X_+(k)}, \quad k \in \mathbb{R}, \tag{1.5}$$

where

$$X_+(k) := G^{1/2}(k) \exp\left(\frac{i}{2} \mathcal{H}[\log G](k) \right), \qquad G(k) := \frac{\lambda\left(k^2 - k_0^2\right)}{\left(\lambda - \hat{K}(k)\right)(k^2+1)}, \quad k \in \mathbb{R} \tag{1.6}$$

with $\mathcal{H}$ denoting Hilbert transform operator as defined in the formulation of the Theorem.

Recalling that, for $x > 0$, $\phi(x) = f(x+a)$, $\lambda = \hat{K}(k_0)$, go back to (1.3), and use Parseval's identity to yield, for $x \in A$,

$$\begin{aligned} f(x) = &- \frac{2\pi i}{\left(\log \hat{K}\right)'(k_0)} \hat{\phi}(-k_0) \left(e^{-2\pi i k_0 (a-x)} \pm e^{-2\pi i k_0 (a+x)} \right) \\ &+ \int_{\mathbb{R}} \hat{\phi}(k)\,\hat{T}(k) \left(e^{-2\pi i k(a-x)} \pm e^{-2\pi i k(a+x)} \right) dk \end{aligned} \tag{1.7}$$

with

$$\hat{T}(k) = \frac{\hat{K}(k)}{\hat{K}(k) - \hat{K}(k_0)} - \frac{2k_0}{\left(k^2 - k_0^2\right)\left(\log \hat{K}\right)'(k_0)}, \quad k \in \mathbb{R}.$$

Since the half-line solution $\phi(x) = f(x+a)$ is a smooth continuation of the solution $f(x)$ on A through $x = a$, the two solutions should match by continuity: $\phi(0^+) = f(a+0^+) = f(a-0^+)$. We thus plug (1.5) into (1.7) and, evaluating both sides at $x = a$, cancel out the non-vanishing factor $\phi(0) = f(a)$ (the property $f(\pm a) \neq 0$ is mentioned in the previous section). This results in a pair of transcendental characteristic equations for determining values of λ corresponding to even or odd set of eigenfunctions depending on the choice of the sign $\pm$ in (1.7). Namely, characteristic equations for even and odd part of the spectrum, respectively, are

$$\frac{i}{2\pi}\int_{\mathbb{R}}\left[\frac{\hat{K}(k)}{\hat{K}(k)-\hat{K}(k_0)}-\frac{2k_0}{\left(k^2-k_0^2\right)\left(\log\hat{K}\right)'(k_0)}\right]\frac{1+e^{-4\pi i k a}}{(k+i)\,X_+(k)}dk=1 + \frac{\left(1+e^{-4\pi i k_0 a}\right)}{(k_0-i)\,X_+(-k_0)\left(\log\hat{K}\right)'(k_0)}, \tag{1.8}$$

$$\frac{i}{2\pi}\int_{\mathbb{R}}\left[\frac{\hat{K}(k)}{\hat{K}(k)-\hat{K}(k_0)}-\frac{2k_0}{\left(k^2-k_0^2\right)\left(\log\hat{K}\right)'(k_0)}\right]\frac{1-e^{-4\pi i k a}}{(k+i)\,X_+(k)}dk=1 + \frac{\left(1-e^{-4\pi i k_0 a}\right)}{(k_0-i)\,X_+(-k_0)\left(\log\hat{K}\right)'(k_0)}. \tag{1.9}$$

These equations are enormously more simple than those obtained by the author in [3] for a particular kernel of the considered class. However, we make another observation that provides even further complexity reduction. We notice that, in some small neighborhood of $x=0$, the integral term is small, i.e. at least $\mathcal{O}\left(1/a^2\right)$ as asymptotics at infinity of Fourier transform of a smooth function (and hence this smallness is eventually related to the smoothness of $\hat{K}$), and hence, in the middle of the interval, the non-integral oscillatory term in (1.7) dominates. On the other hand, we know that all eigenfunctions are real-valued (up to a choice of normalization). This imposes a restriction on the complex phase of the constant in front of the oscillatory function which is either $\cos(2\pi k_0 x)$ (in case of even eigenfunctions) or $\sin(2\pi k_0 x)$ (in case of odd eigenfunctions). Employing (1.6), (1.5), we thus deduce the following versions of characteristic equations to be solved for $k_0>0$ (or, equivalently, for $\lambda=\hat{K}(k_0)$)

$$2\pi k_0 a-\arctan\frac{1}{k_0}-\frac{1}{2}\mathcal{H}\left[\log G\right](k_0)=\pi m,\quad m\in\mathbb{Z},\quad \text{(even eigenfunctions)} \tag{1.10}$$

$$2\pi k_0 a-\arctan\frac{1}{k_0}-\frac{1}{2}\mathcal{H}\left[\log G\right](k_0)=\pi\left(m+\frac{1}{2}\right),\quad m\in\mathbb{Z}\quad \text{(odd eigenfunctions)} \tag{1.11}$$

with the large-value λ corresponding to the solutions with small positive values of k_0.

For each solution k_0 of transcendental equations (1.8)-(1.9) or (1.10)-(1.11), the corresponding eigenfunction (up to a normalization constant) would be given by (1.7) (with the choice of the appropriate sign $\pm$). That is, we have explicitly, up to a multiplicative normalization constant, even eigenfunctions:

$$f(x)=-\frac{2e^{-2\pi i k_0 a}}{(k_0-i)\,X_+(-k_0)\left(\log\hat{K}\right)'(k_0)}\cos(2\pi k_0 x) + \frac{i}{\pi}\int_{\mathbb{R}}\frac{e^{-2\pi i k a}}{(k+i)\,X_+(k)}\left[\frac{\hat{K}(k)}{\hat{K}(k)-\hat{K}(k_0)}-\frac{2k_0}{\left(k^2-k_0^2\right)\left(\log\hat{K}\right)'(k_0)}\right]\cos(2\pi k x)\,dk, \tag{1.12}$$

odd eigenfunctions:

$$
\begin{aligned}
f(x) = &-\frac{2ie^{-2\pi i k_0 a}}{(k_0 - i)\, X_+(-k_0) \left(\log \hat{K}\right)'(k_0)} \sin(2\pi k_0 x) \\
&-\frac{1}{\pi}\int_{\mathbb{R}} \frac{e^{-2\pi i k a}}{(k+i)\, X_+(k)} \left[\frac{\hat{K}(k)}{\hat{K}(k) - \hat{K}(k_0)} - \frac{2k_0}{\left(k^2 - k_0^2\right)\left(\log \hat{K}\right)'(k_0)}\right] \sin(2\pi k x)\, dk,
\end{aligned}
\tag{1.13}
$$

where X_+ and G are given in terms of $\hat{K}$ and k_0 by (1.6). Note that the pre-trigonometric factors in the first term of (1.12)-(1.13) simplify due to characteristic equations (1.10)-(1.11) leading to the form of eigenfunctions as in the statement of Theorem.

□

1.3 Solution for small A

Theoretical works where the case of small interval is considered are rare but do exist (e.g. see [11]) for the Wiener-Hopf treatment corresponding to a class of inhomogeneous integral equations).

It is convenient to start by rescaling. We denote $\varphi(x) := f(ax)$, $\eta = \lambda/a$ and rewrite equation (1.2) as

$$
\int_{-1}^{1} K(a(x-t))\, \varphi(t)\, dt = \eta \varphi(x), \quad x \in (-1, 1). \tag{1.14}
$$

Since the interval A is small, a kernel function can be well approximated by a span of very few elementary basis functions such as monomials in case of Taylor expansion $K(ax) = \sum_{m=0}^{M} \frac{K^{(m)}(0)}{m!}(ax)^m$. This approximation, however, has natural limitations since the resulting integral operator is of finite rank and hence cannot have more than $M+1$ eigenfunctions. Clearly, to reproduce rich enough structure of the original operator, M has to be taken very large.

The idea of our approach is still to take advantage of the fact that only a couple of degrees of freedom are needed for decent approximation of a kernel function in the neighborhood of the origin (i.e. on the small interval A), and select a class of functions which is, on one hand, solvable and, on the other hand, corresponds to a non-degenerate compact operator, that is an operator with infinite dimensional range.

In particular, as such approximant class, we consider, for $x \in (-1, 1)$, a family of functions

$$
K_{b,c}(x) := \frac{\sin cx}{\sinh bx}, \quad b, c \in \mathbb{R} \cup i\mathbb{R}. \tag{1.15}
$$

All integral operators with kernels from the class (1.15) are very special as they remarkably commute with a differential operator $-\frac{d}{dx}\left(1 - \frac{\sinh^2 bx}{\sinh^2 b}\right)\frac{d}{dx} + \left(b^2 + c^2\right)\frac{\sinh^2 bx}{\sinh^2 b}$ which is an extremely rare situation [10, 32]. Due to such commutation, the integral and differential operators possess a common set of eigenfunctions, and since the spectrum is simple, the original problem is equivalent to solving a boundary-value ODE problem. Namely, the eigenfunctions are regular solutions of

$$
-\left(\left(1 - \frac{\sinh^2 bx}{\sinh^2 b}\right)\varphi'(x)\right)' + \left(b^2 + c^2\right)\frac{\sinh^2 bx}{\sinh^2 b}\varphi(x) = \mu \varphi(x), \quad x \in (-1, 1). \tag{1.16}
$$

The condition of finiteness of solution at the endpoints $x = \pm 1$ restricts the values of eigenparameter $\mu \in \mathbb{R}$ to an infinite discrete set. Note that despite sharing the same set of eigenfunctions, eigenvalues $(\mu_n)_{n=1}^{\infty}$ of differential operator are different from eigenvalues $(\lambda_n)_n^{\infty}$ of the original problem (1.2). The latter can be recovered from (1.14) in a numerically stable way as

$$\lambda_n = \frac{a \int_{-1}^{1} \varphi_n(x) \int_{-1}^{1} K(a(x-t)) \varphi_n(t)\, dtdx}{\|\varphi_n\|_{L^2(-1,1)}^2}, \quad n \in \mathbb{N}_+.$$

Let us now consider a particular instance of the described approximation procedure.

In common practical applications a convolution kernel has a "hump" structure, that is a symmetric function concentrated near the origin. Such behavior can be approximated by hyperbolic secant function $\operatorname{sech} x \equiv \frac{1}{\cosh x}$ belonging to the considered family (1.15) with $b = 2$, $c = i$. In other words, instead of (1.14), we now consider its approximate version

$$\int_{-1}^{1} \operatorname{sech}(a(x-t))\, \varphi(t)\, dt = \eta \varphi(x), \quad x \in (-1, 1).$$

The corresponding ODE is then

$$-\left(\left(1 - \frac{\sinh^2 ax}{\sinh^2 a}\right) \varphi'(x)\right)' + 3a^2 \frac{\sinh^2 ax}{\sinh^2 a} \varphi(x) = \mu \varphi(x), \quad x \in (-1, 1).$$

Upon further approximation for $a \ll 1$, this ODE becomes

$$\left(\left(1 - x^2\right) \varphi'(x)\right)' + \left(\mu - 3a^2 x^2\right) \varphi(x) = 0, \quad x \in (-1, 1)..$$

which is a well-studied equation [24] whose solutions are bounded on $[-1, 1]$ only for special values $\mu_n = \chi_{n-1}\left(\sqrt{3}a\right)$, $n \in \mathbb{N}_+$ and are expressible as prolate spheroidal wave functions $\varphi_n(x) = S_{0(n-1)}\left(\sqrt{3}a, x\right)$, $n \in \mathbb{N}_+$ (see [27] for the notation).

The efficiency of such approximation was demonstrated on the Poisson kernel $K(x) = \frac{1}{1+x^2}$ in [3]. This success might be related to the fact that Fourier transform of both kernels have, in some sense, similar structure (in Fourier domain), but this has to be investigated further.

1.4 Conclusion

We have described new methods for obtaining asymptotical solutions of convolution integral equations on a finite interval focussing on a notoriously difficult spectral version of such equation.

When the interval is large, under assumption of even parity of the kernel function, as well as mild assumptions on its decay and regularity (and with additional simplifying assumptions on its Fourier transform: positivity and strictly monotonic decay from of the origin), we have found a closed-form approximation to eigenfunctions of the integral operator and formulated characteristic equations for eigenvalues of odd and even part of the spectrum. We also pointed out an alternative form of such transcendental characteristic equations which provides tremendous simplification in spectrum calculation with respect to previously obtained results for particular instances of the kernel functions of the considered class. Structural simplicity of eigenfunctions is also remarkable: away from the interval endpoints they are essentially sine and cosine functions of special frequencies whereas a more complicated integral term dominates behavior near the endpoints. Such behavior is not unexpected as it has been observed for a more restrictive class of kernel functions, namely those rapidly decaying at infinity [12, 16].

While the situation for small intervals is theoretically and practically less interesting, we attempt to make some contribution by advocating the use of non-orthodox kernel approximation. Namely, instead of classical finite-rank operator approximation by polynomials or trigonometric function, we propose to approximate kernel function by a function from a 2-parameter function family that corresponds to a set of compact self-adjoint operators posessing an infinite set of eigenfunctions. The advantage of the selected function family is that it allows reducing the integral equation to an ODE problem, and in some cases solutions are readily available in terms of known special functions. In view of rather standard behavior of general smooth kernel function near the origin, a suitable approximant from the considered family would suffice to reproduce the main features of the integral operator better than its low dimensional spectral approximation.

In this work, we have only outlined a solution procedure focussing on constructive aspects and providing intuitive justification whereas details, rigorous proofs and study of approximation properties are not given here. For example, from a structure of the neglected term in our large-interval approximation (which we did not even write), it is clear that the faster K decays at infinity, the better is the approximation, but it is necessary to quantify this precisely and propagate this error of approximation of auxiliary problem into the estimates of the final error terms $\delta_{n,a}^{(e)}$, $\delta_{n,a}^{(o)}$, $\left\|\mathcal{E}_{n,a}^{(e)}\right\|_{L^\infty(A)}$, $\left\|\mathcal{E}_{n,a}^{(o)}\right\|_{L^\infty(A)}$ appearing in the formulation of Theorem 1. The errors in the small interval case should be estimated as well.

We are confident that presented results can be extended (at the expense of more cumbersome expressions) to even more general class of kernels encompassing those whose Fourier transforms do not necessarily decay strictly monotonically. Additonally, we expect that some of the assumptions can be tightened (e.g. assuming $\hat{K} \in L^1(\mathbb{R})$ instead of $K \in C^1(\mathbb{R})$).

Detailed account of study of these aspects will be published separately elsewhere [25].

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