MAGNETISATION MOMENT OF A BOUNDED 3D SAMPLE: ASYMPTOTIC RECOVERY FROM PLANAR MEASUREMENTS ON A LARGE DISK

DMITRY PONOMAREV [1,2]

ABSTRACT. We consider the problem of reconstruction of the net moment (vector of the overall magnetisation) of a sample from partial data of the magnetic field. Namely, motivated by a concrete experimental set-up, we deal with a situation when the magnetic field is measured on a portion of the plane in vicinity of the sample and only one (normal to the plane) component of the field is available. Under assumption that the measurement area is a sufficiently large disk (lying in a horizontal plane above the sample), we obtain a set of estimates for the components of the net moment vector with the accuracy which improves asymptotically with the increase of the measurement disk radius. Compared to our previous preliminary results, the asymptotic formulas are now rigorously justified and higher-order estimates are derived. Moreover, the presented approach, based on an appropriate splitting in the Fourier domain and estimates of oscillatory integrals (involving both small and large parameters), elucidates the derivation of asymptotic estimates of an arbitrary order, a possibility that was previously unclear. The obtained results are illustrated numerically and their robustness with respect to noise is discussed. The proposed methodology should be applicable to other magnetic and gravimetric problems with planar measurements.

1. INTRODUCTION

Constant advances in magnetometry allow measurements of magnetic fields of very low intensities with high spatial resolution. In particular, this opens new horizons in paleomagnetic contexts. Ancient rocks and meteorites possess remanent magnetisation and thus might preserve valuable records of a past magnetic activity on Earth and other planets, asteroids and satellites. Extraction of this relict magnetic information is a lucrative but challenging task. Deducing magnetisation of a geosample hinges on effective processing of the measurements of the magnetic field available in a nearest neighbourhood of the sample since the informative part of the field further away is very weak and significantly deteriorated by noise. In particular set-ups of scanning SQUID (Superconducting Quantum Interference Device) magnetometer or QDM (Quantum Diamond Microscope), measurements are available in a planar area above the sample, in a close vicinity of it, and such measurements typically feature only one component of the magnetic field. This is in contrast with more common settings that deal with magnetic fields of higher intensity and hence could, on a methodological level, rely on the classical dipolar approximation of a sample valid in a far-away region.

In the present work, we are concerned with recovery of the overall magnetisation (the so-called net moment) of a sample rather than dealing with reconstruction of the entire magnetisation distribution. While both inverse problems are known to be ill-posed (as an inverse source problem for elliptic partial-differential equations) due to the lack of continuous dependence of their solution on the input data (magnetic field measurements), the problem of reconstruction of full magnetisation distribution additionally lacks uniqueness of the solution in view of presence of invisible (or "silent") sources, i.e., magnetisations that do not produce magnetic field, see [4]. However, as it was shown in [2] for planar (thin-plate) magnetisation distributions, and as follows from the general Helmholtz decomposition [7], compactly supported invisible sources do not contribute to the net moment. This statement fixes the non-uniqueness issue and makes the problem of net moment recovery a feasible task. In theory, this problem is even solvable in a closed form when measurements are available on the entire plane above the sample. In reality, however, the measurements are very limited and corrupted by the presence of noise which dominates the signal in distant regions. Therefore, we arrive at the problem of estimating the net moment of a sample from a magnetic field component available on some portion of the plane in proximity of the sample. We shall assume that this portion of the plane is sufficiently large, for otherwise the instability of the problem, due to the already mentioned inherent ill-posedness, will be even more severe. On the other hand, this assumption on the large size of the measurement area allows us to obtain explicit ready-to-use formulas.

We do not intend here to provide neither physical nor mathematical description of the problem in any detailed fashion. Instead, we refer the reader to the set of previous publications [4, 13, 2, 6, 15, 10, 26, 14] and briefly

---

[1]FACTAS team, Centre Inria d'Université Côte d'Azur, France

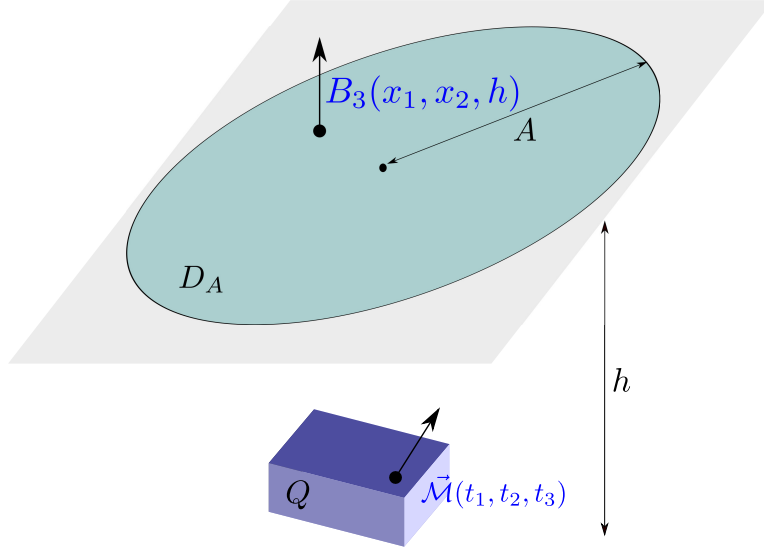[2]Contact: dmitry.ponomarev@inria.fr

FIGURE 1.1. Schematic illustration of geometry of the problem arising from experimental set-up in the Paleomagnetism lab at EAPS, MIT (USA). B_3 is a vertical component of the magnetic field measured over the horizontal disk D_A at some distance h from a sample Q with the magnetisation distribution $\vec{\mathcal{M}}$.

introduce basic concepts that allow us to be more specific in describing our main result and comparing it with relevant works.

We should mention that the present setting is somewhat classical for problems in magnetometry or gravimetry (i.e., detection of underground anomalous region from surveillance data), with a particularity here that only one component is available for direct measurements. The obtained results hence should be useful beyond the laboratory context of rock paleomagnetism, and the proposed methodology is clearly extendable to other particular set-ups.

We assume that the magnetic sample is described by a compactly supported vector distribution

$$\vec{\mathcal{M}}(\vec{x}) \equiv (\mathcal{M}_1(x_1, x_2, x_3), \mathcal{M}_2(x_1, x_2, x_3), \mathcal{M}_3(x_1, x_2, x_3))^T, \quad \mathrm{supp}\, \vec{\mathcal{M}} \subset Q,$$

with some bounded set $Q \subset \mathbb{R}^3$.

The relation between the unknown magnetisation distribution $\vec{\mathcal{M}}$ and the vertical component of the produced magnetic field is

$$(1.1) \qquad B_3(\vec{x}) = \frac{\partial}{\partial x_3} \iiint_Q \frac{1}{4\pi |\vec{x} - \vec{t}|} \nabla \cdot \vec{\mathcal{M}}(\vec{t}) \, \mathrm{d}^3 t, \quad \vec{x} \in \mathbb{R}^3 \backslash \overline{Q}.$$

This latter quantity is experimentally measured on a portion of the horizontal plane at height $x_3 = h$ for some constant $h > 0$ such that this plane does not intersect Q, and it can be equivalently written as

$$(1.2)\ B_3(\mathbf{x}, h) = \iiint_Q \frac{3(h - t_3)[\mathcal{M}_1(\mathbf{t}, t_3)(x_1 - t_1) + \mathcal{M}_2(\mathbf{t}, t_3)(x_2 - t_2)] + \mathcal{M}_3(\mathbf{t}, t_3)(2(h - t_3)^2 - |\mathbf{x} - \mathbf{t}|^2)}{4\pi (|\mathbf{x} - \mathbf{t}|^2 + (h - t_3)^2)^{5/2}} \mathrm{d}^3 t.$$

Here and onwards, we employ bold symbols to denote $\mathbb{R}^2$ vectors, e.g., $\mathbf{x} \equiv (x_1, x_2)^T$ while using $\vec{x} \equiv (x_1, x_2, x_3)^T$ for $\mathbb{R}^3$ vectors. When $\vec{\mathcal{M}}$ is a distribution, the integral on the right-hand side of (1.2) should be understood as a sum of three terms, each is the duality pairing of a compactly supported scalar distribution $\mathcal{M}_1$, $\mathcal{M}_2$ or $\mathcal{M}_3$ with the corresponding smooth function on $\mathbb{R}^3$ (note that since the measurement plane does not intersect Q, the denominator is bounded away from zero and thus no singularities arise).

The geometry of the described setting is schematically shown in Figure 1.1. This corresponds, up to a truncation of rectangular magnetic field map, to an experimental set-up of the Paleomagnetism lab at EAPS (Earth, Atmospheric and Planetary Sciences) department, MIT (Massachusetts Institute of Technology), involving a SQUID magnetometer, see [26]. Moreover, with an extra preprocessing step of the field data, this also extends to the QDM magnetometer set-up used in the Paleomagnetism lab at Harvard University [10].

A quantity of basic physical interest is the net magnetisation moment:

$$(1.3) \qquad \vec{m} \equiv (m_1, m_2, m_2)^T := \iiint_Q \vec{\mathcal{M}}(\vec{x}) \, \mathrm{d}^3 x.$$

This is a constant $\mathbb{R}^3$ vector equal to the overall magnetisation of the sample which can be determined uniquely, unlike the full magnetisation distribution $\vec{\mathcal{M}}$ which essentially enters B_3 through $\nabla \cdot \vec{\mathcal{M}}$ and hence can be changed by any divergence-free vector field without altering B_3, see (1.1) and discussions in [4]. In what follows, we may intechangeably use slightly different names for $\vec{m}$: net moment, net magnetisation, zeroth-order algebraic moment of a magnetisation distribution. When $\vec{\mathcal{M}}$ is not a regular function but a distribution, the integral in (1.3) should be understood componentwise as the duality pairing of a compactly supported distribution with the constant function 1 on $\mathbb{R}^3$: $m_j = \langle \mathcal{M}_j, 1 \rangle$, $j \in \{1, 2, 3\}$.

The present paper is dedicated to the explicit asymptotic estimation of the net magnetisation moment $\vec{m}$ from measurements of the magnetic field B_3 over a large planar area. In particular, as the measurement area, we have taken the horizontal disk D_A of radius A located at height h above the magnetic sample. We formally derive and prove that each of the quantities m_1, m_2 and m_3 can be estimated by means of integration of the measured magnetic field data B_3 on D_A against an appropriate function. We obtain several of such estimates depending on the asymptotic order (up to order 5 for m_1, m_2, and up to order 4 for m_3), though this process could be continued to obtain even higher order estimates. As it will be discussed, higher-order estimates do not make those of lower order redundant because of different level of sensitivity to noise which is unavoidable in any realistic setting.

It should be mentioned that even though recovery of the net moment of a sample, our primary concern, is an important practical problem on its own right, it can also serve, under appropriate assumptions, as a preliminary step for the full magnetisation inversion (i.e., finding a magnetisation distribution $\vec{\mathcal{M}}$ that would be consistent with the measured data B_3 according to (1.2)). Indeed, while it is unrealistic to retrieve three generally independent magnetisation components (functions or, more generally, distributions), $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, from the partial knowledge of only one function B_3 (see (1.2)) without additional assumptions, the problem simplifies significantly for a class of samples which are unidirectionally magnetised (i.e., when magnetisation direction does not change throughout the sample but its magnitude does). Since quantifying the net moment of a sample implies a definite magnetisation direction, it is thus an essential element in this complete reconstruction procedure, see [13].

Note that the obtained estimates may be used either directly on the data when the original measurement area is sufficiently large compared to the localisation of the magnetisation in the sample and the lift-off distance (scanning height h), or after a preliminary application of the field extrapolation, see [22].

We stress that our analytical expressions for the asymptotic estimates are original, with only partial analogs in other works when translated into the current context. According to [20], for extended finite magnetic sources, net moment estimates in terms of the measured field have first appeared in Helbig's work [12] and later rediscovered by Clark and Schmidt, see, e.g., [23]. When truncated to a finite measurement area, these estimates correspond to our first-order asymptotic formulas for tangential components of net moment, but without a mention of the order of the asymptotic convergence, let alone a possibility of obtaining their more precise higher-order counterparts. Note that, in our context, since only B_3 component of the field is measured, those previous estimates do not provide means to estimate the normal net moment component m_3. Indeed, an estimate of m_3 was provided only in terms of either B_1 or B_2 components. When measurements of these two field components are not available, one can potentially argue that it is still possible to use the Helbig's formulas after reconstructing the missing components of the field from the knowledge of B_3 alone using, for example, [11] and [8, Ch. 12]. However, this is possible only to accomplish in a stable way only if B_3 is known on the entire plane. In view of this, we believe that an explicit estimate of m_3 from knowledge of B_3 alone on a finite region for the first time was proposed only in [21, 5, 3]. It should be noted that results in [3] are also applicable for measurement regions different from disk, namely, for those of rectangular and diamond shapes. We take advantage of this comparison of results to mention and to correct a couple of issues in [5]. First of all, there is a typo in the m_3 formula obvious when compared to (2.4) or [21, Thm 3.1.1]: in eq. [5, Eq. (5)], the factor A should be in the numerator rather than the denominator. Moreover, the numerical illustration of the m_3 estimate, in case of noisy data, showed a divergent behavior with the growth of the measurement disk, and this linearly growing trend was proposed to be removed by an appropriate postprocessing. It turns out that this undesirable behavior was merely a consequence of the numerical implementation (Simpson's quadrature rule should have been avoided due to the low regularity of the noisy field) and not the asymptotic formula itself, as demonstrated in the present work; the same is also true for higher-order estimates of m_3. The drawback of finiteness of the measurement region for the application of Helbig's integral formulas has been recognised a while ago and its influence has been analysed, see, e.g., [17]. One possibility to deal with this was to choose and correct a suitable integration window so that the consistency of Helbig's integrals is respected (since some integrals of the field data must be identically zero), see [20]. Another approach consists in replacing the missing field data by the field of a fictious dipole whose location and moment are estimated iteratively from appropriate integrals of the available data [1, 9]. A similar iterative strategy [18] is based on the multipolar expansion of order 2 (involving quadrupole moment tensor). In a recent work [16], a multipolar approach of arbitrary order has been used to fit the measured data

furnishing the net moment as first coefficients of the multipolar expansion. This shows good results but requires a potentially delicate parameter tunning (such as choice of the origin and the order of the expansion) as a part of the regularisation process. A totally different method which does not theoretically require a large measurement region is that based on a bounded extremal problem for finding best-possible linear estimators [2]. These estimators are auxiliary functions whose integrals against the measured data furnish the components of the net moment. Finding such estimators in a particular functional class requires solving an integro-differential equation which is a numerically laborous task. While formally applicable for any size and position of the measurement region with respect to the magnetic source, a practical sensitivity of that approach to perturbations of the measured data must increase enormously when the measurement area is not sufficiently large or well-located with respect to the source. Moreover, this approach is restricted to planar and regular (square-integrable) sources at a known height (depth).

The present approach is applicable for one-component field measurements. It has high accuracy already for relatively small measurement area due to the higher-order asymptotic estimates (which, unlike Helbig's integrals, include also estimates for m_3 from B_3). The approach is easy to implement numerically due to the explicit nature of the estimates. It is applicable for bounded volumetric magnetic sources which are not necessarily regular (they may be even compactly supported distributions, not just square-integrable functions). Moreover, the approach does not require the knowledge of the depth of the source (as all the estimates are seen to be independent of the height parameter h which must only satisfy a certain *a priori* assumption).

The structure of the paper is as follows. Section 2 presents the main results of the paper formulated as Theorem 2.1 and discusses the limitations of their applicability. Section 3 has a twofold goal. First, it is meant to show how one idea based on straightforward Fourier analysis can yield the simplest version of asymptotic net moment estimates for both tangential and normal components. Second, that section illustrates that, by means of a careful asymptotic analysis, the explicit estimates can be not only proved rigorously but also extended to higher orders. Hence, this material exactly constitutes the proof of Theorem 2.1. Then, in Section 4, we illustrate the results numerically on the case where the magnetisation has a singular support (a collection of dipoles is modelled by magnetisation distribution that is a sum of Dirac delta functions) and deal with some practical aspects of the obtained estimates. Finally, we conclude with Section 5 summarising the work, discussing the obtained results and outlining potential further research directions.

2. Main results

Let $\mathcal{E}'\left(\mathbb{R}^3\right)$ be the space of compactly supported distributions on $\mathbb{R}^3$, i.e., the linear functionals on smooth functions $C^\infty\left(\mathbb{R}^3\right)$, see [25, Sect. 6.1].

Denote $D_A := \left\{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < A\right\}$, the disk of radius A centered at the origin $\mathbf{x} = \mathbf{0}$.

The main result of this work is summarised in the following theorem.

Theorem 2.1. *Assume that $\vec{\mathcal{M}} \in \left[\mathcal{E}'\left(\mathbb{R}^3\right)\right]^3$ with $\operatorname{supp} \vec{\mathcal{M}} \subset Q$ for a bounded set $Q \subset \mathbb{R}^3$. Suppose that the values of the vertical component of the magnetic field B_3 (related to $\vec{\mathcal{M}}$ by means of (1.2)) are known on the horizontal disk $D_A \times \{x_3 = h\}$ that does not intersect Q and whose radius A is sufficiently large so that the following inequality holds:*

$$(2.1) \qquad \sup_{\vec{t} \in Q,\, \mathbf{x} \in \mathbb{R}^2 \setminus D_A} \left| \frac{t_1^2 + t_2^2 + (h - t_3)^2}{x_1^2 + x_2^2} - 2\frac{x_1 t_1 + x_2 t_2}{x_1^2 + x_2^2} \right| < 1.$$

Then, the components of the net moment vector (1.3) can be asymptotically estimated with different orders of accuracy as follows.

First-order estimates:

$$(2.2) \qquad m_j = 2 \iint_{D_A} x_j B_3\left(\mathbf{x}, h\right) \mathrm{d}^2 x + \mathcal{O}\left(\frac{1}{A}\right), \qquad j \in \{1, 2\}.$$

Second-order estimates:

$$(2.3) \qquad m_j = 2 \iint_{D_A} \left(1 + \frac{4x_j^2}{3A^2}\right) x_j B_3\left(\mathbf{x}, h\right) \mathrm{d}^2 x + \mathcal{O}\left(\frac{1}{A^2}\right), \qquad j \in \{1, 2\},$$

$$(2.4) \qquad m_3 = 2A \iint_{D_A} B_3\left(\mathbf{x}, h\right) \mathrm{d}^2 x + \mathcal{O}\left(\frac{1}{A^2}\right).$$

Third-order estimates:

$$(2.5) \qquad m_j = \frac{2}{5} \iint_{D_A} \left[5 + 24\left(\frac{x_j}{A}\right)^4\right] x_j B_3\left(\mathbf{x}, h\right) \mathrm{d}^2 x + \mathcal{O}\left(\frac{1}{A^3}\right), \qquad j \in \{1, 2\},$$

$$(2.6) \qquad m_3 = \frac{A}{4} \iint_{D_A} \left[5 + 40 \left(\frac{x_j}{A} \right)^4 - 128 \left(\frac{x_j}{A} \right)^6 \right] B_3 \left(\mathbf{x}, h \right) \mathrm{d}^2 x + \mathcal{O} \left(\frac{1}{A^3} \right), \qquad j \in \{1, 2\}.$$

Fourth-order estimates:

$$(2.7)$$
$$m_j = \frac{2}{105} \iint_{D_A} \left[105 - 2016 \left(\frac{x_j}{A} \right)^4 + 19200 \left(\frac{x_j}{A} \right)^6 - 22400 \left(\frac{x_j}{A} \right)^8 \right] x_j B_3 \left(\mathbf{x}, h \right) \mathrm{d}^2 x + \mathcal{O} \left(\frac{1}{A^4} \right), \qquad j \in \{1, 2\},$$

$$(2.8) \qquad m_3 = \frac{A}{24} \iint_{D_A} \left[35 + 1792 \left(\frac{x_j}{A} \right)^6 - 3200 \left(\frac{x_j}{A} \right)^8 \right] B_3 \left(\mathbf{x}, h \right) \mathrm{d}^2 x + \mathcal{O} \left(\frac{1}{A^4} \right), \qquad j \in \{1, 2\}.$$

Fifth-order estimates:

$$(2.9)$$
$$m_j = \frac{2}{231} \iint_{D_A} \left[231 - 52800 \left(\frac{x_j}{A} \right)^6 + 246400 \left(\frac{x_j}{A} \right)^8 - 225792 \left(\frac{x_j}{A} \right)^{10} \right] x_j B_3 \left(\mathbf{x}, h \right) \mathrm{d}^2 x + \mathcal{O} \left(\frac{1}{A^5} \right), \qquad j \in \{1, 2\}.$$

Remark 2.2. Here and onwards (except numerics in Section 4), for the sake of simplicity, we have assumed the system of physical units such that the constant of magnetic permeability of vacuum μ_0 is 1. In general, the right-hand sides of expression (1.2) should have the factor μ_0, and, in Si units, $\mu_0 = 4\pi \cdot 10^{-7}$ N / A^2. Consequently, the right-hand sides of all the formulas (2.2)–(2.9) should, in principle, have the factor $1/\mu_0$.

Remark 2.3. It is not difficult to see that condition (2.1) can be replaced with

$$(2.10) \qquad \inf_{\vec{t} \in Q, \ \mathbf{x} \in \mathbb{R}^2 \backslash D_A} \frac{(x_1 - t_1)^2 + (x_2 - t_2)^2}{2 \left(t_1^2 + t_2^2 \right) + \left(h - t_3 \right)^2} > 1.$$

Indeed, (2.1) is obtained from (see (3.23))

$$-1 < \frac{t_1^2 + t_2^2 + \left(h - t_3 \right)^2}{x_1^2 + x_2^2} - 2 \frac{x_1 t_1 + x_2 t_2}{x_1^2 + x_2^2} < 1, \qquad \vec{t} \in Q, \ \mathbf{x} \in \mathbb{R}^2 \backslash D_A.$$

Here, the validity of the left inequality is trivial since

$$(x_1 - t_1)^2 + (x_2 - t_2)^2 + (h - t_3)^2 > 0, \qquad \vec{t} \in Q, \ \mathbf{x} \in \mathbb{R}^2 \backslash D_A,$$

whereas the right inequality can be rewritten as

$$x_1^2 + x_2^2 + 2 (x_1 t_1 + x_2 t_2) + t_1^2 + t_2^2 > 2 \left(t_1^2 + t_2^2 \right) + \left(h - t_3 \right)^2, \qquad \vec{t} \in Q, \ \mathbf{x} \in \mathbb{R}^2 \backslash D_A.$$

By the symmetry of the area $\mathbb{R}^2 \backslash D_A$, we can change the signs in front of x_1 and x_2. Recognising the complete square on the left-hand side and dividing over the positive expression from the right-hand side, we arrive exactly at the fraction appearing in (2.10) and it only rests to take the infimum.

Finally, let us point out a simplification occuring in the common setting (e.g., when Q is a minimal rectangular parallelipiped or a ball containing the magnetisation support located under the measurement device). Namely, if Q is a bounded connected set whose lowest points lie at $x_3 = 0$ plane and whose horizontal projection Q_{12} contains the center $\mathbf{x} = \mathbf{0}$ of the measurement disk D_A, then inequality (2.10) can be ensured by imposing a stricter but geometrically simpler condition

$$(2.11) \qquad \frac{2 \left[\mathrm{dist} \left(\partial D_A, Q_{12} \right) \right]^2}{\left(\mathrm{diam} \ Q_{12} \right)^2 + 2h^2} > 1,$$

where diam Q_{12} denotes the diameter of Q_{12} and dist $(\partial D_A, Q_{12})$ is the distance between the boundary of the measurement disk and Q_{12}. Indeed, by the above assumption on the location of Q and choice of the origin, we have $t_1^2 + t_2^2 \leq (\mathrm{diam} \ Q_{12})^2 / 4$, $(h - t_3)^2 \leq h^2$ for $\vec{t} \in Q$, and hence the inequality

$$\inf_{\vec{t} \in Q, \ \mathbf{x} \in \mathbb{R}^2 \backslash D_A} \frac{(x_1 - t_1)^2 + (x_2 - t_2)^2}{2 \left(t_1^2 + t_2^2 \right) + \left(h - t_3 \right)^2} \geq \frac{1}{\left(\mathrm{diam} \ Q_{12} \right)^2 / 2 + h^2} \inf_{\vec{t} \in Q, \ \mathbf{x} \in \mathbb{R}^2 \backslash D_A} \left[(x_1 - t_1)^2 + (x_2 - t_2)^2 \right]$$
$$= \frac{2 \left[\mathrm{dist} \left(\partial D_A, Q_{12} \right) \right]^2}{\left(\mathrm{diam} \ Q_{12} \right)^2 + 2h^2}$$

means that (2.11) would imply (2.10).

Note that condition (2.11) specific to our asymptotic approach is actually quite natural and generally consistent with the geometric setting in which a good (reasonably stable) inversion can be expected since the Poisson transformation lying at the heart of the integral operator in (1.2) is known to rapidly spread away the source information, see also [17].

3. PROOF OF THEOREM 2.1

3.1. **Some notation and preparatory transformations.** Before we proceed with deriving rigorously formulas (2.2)–(2.9) and thus proving Theorem 2.1, let us introduce some useful notations. We shall use the following notational shortcut for the integral of the magnetisation distribution against monomials

$$(3.1) \qquad \left\langle x_1^{j_1} x_2^{j_2} x_3^{j_3} M_n \right\rangle := \iiint_Q x_1^{j_1} x_2^{j_2} x_3^{j_3} \mathcal{M}_n\left(\vec{x}\right) \mathrm{d}^3 x, \qquad n \in \{1,2,3\}, \quad j_1, j_2, j_3 \in \mathbb{N}_0,$$

where we denoted $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, the set of natural numbers with zero. Following this convention, for the sake of brevity, we may also write, for example,

$$\left\langle (h - x_3) M_n \right\rangle := h m_n - \left\langle x_3 M_n \right\rangle = \iiint_Q (h - x_3) \mathcal{M}_n\left(\vec{x}\right) \mathrm{d}^3 x, \qquad n \in \{1, 2, 3\}.$$

As before, we note that, when magnetisation $\vec{\mathcal{M}}$ is not a function but a compactly supported distribution, the integrals above should be understood as the duality pairings between $\mathcal{M}_n \in \mathcal{E}'\left(\mathbb{R}^3\right)$ and $x_1^{j_1} x_2^{j_2} x_3^{j_3} \in C^\infty\left(\mathbb{R}^3\right)$: $\iiint_Q x_1^{j_1} x_2^{j_2} x_3^{j_3} \mathcal{M}_n\left(\vec{x}\right) \mathrm{d}^3 x \equiv \left\langle \mathcal{M}_n, x_1^{j_1} x_2^{j_2} x_3^{j_3} \right\rangle$ with $j_1, j_2, j_3 \in \mathbb{N}_0$, $n \in \{1, 2, 3\}$.

Let us $\hat{\cdot}$ denote the two-dimensional Fourier transform which, by our convention, is defined as

$$\hat{f}\left(\mathbf{k}\right) \equiv \mathcal{F}\left[f\right]\left(\mathbf{k}\right) := \iint_{\mathbb{R}^2} e^{2\pi i \mathbf{k} \cdot \mathbf{x}} f\left(\mathbf{x}\right) \mathrm{d}^2 x,$$

where $i = \sqrt{-1}$ stands for the imaginary unit, and $\mathbf{k} \cdot \mathbf{x} = k_1 x_1 + k_2 x_2$ is the Euclidean inner product. With this definition, the differentiation and convolution properties of Fourier transform have the form

$$(3.2) \qquad \mathcal{F}\left[\partial_{x_j} f\right]\left(\mathbf{k}\right) = -2\pi i k_j \hat{f}\left(\mathbf{k}\right), \quad \mathcal{F}\left[x_j f\right]\left(\mathbf{k}\right) = \frac{1}{2\pi i} \partial_{k_j} \hat{f}\left(\mathbf{k}\right), \qquad j \in \{1, 2\},$$

$$(3.3) \qquad \mathcal{F}\left[f \star g\right]\left(\mathbf{k}\right) := \iint_{\mathbb{R}^2} e^{2\pi i \mathbf{k} \cdot \mathbf{x}} \iint_{\mathbb{R}^2} f\left(\mathbf{x} - \mathbf{t}\right) g\left(\mathbf{t}\right) \mathrm{d}^2 t \, \mathrm{d}^2 x = \hat{f}\left(\mathbf{k}\right) \hat{g}\left(\mathbf{k}\right).$$

We also note that the Fourier transform of the two-dimensional Poisson kernel is well-known (see, e.g., [25, Sect. 4.2]), that is, for any $H > 0$, we have

$$(3.4) \qquad \mathcal{F}\left[\frac{H}{2\pi \left(\left|\mathbf{x}\right|^2 + H^2\right)^{3/2}}\right]\left(\mathbf{k}\right) = e^{-2\pi H |\mathbf{k}|}, \quad \mathbf{k} \in \mathbb{R}^2.$$

Let us now rewrite (1.2) as

$$(3.5) \qquad B_3\left(\mathbf{x}, h\right) = -\frac{1}{4\pi} \iiint_Q \left[\left(\mathcal{M}_1\left(\mathbf{t}, t_3\right) \frac{\partial}{\partial x_1} + \mathcal{M}_2\left(\mathbf{t}, t_3\right) \frac{\partial}{\partial x_2} \right) \frac{h - t_3}{\left(\left|\mathbf{x} - \mathbf{t}\right|^2 + \left(h - t_3\right)^2\right)^{3/2}} \right.$$

$$\left. + \mathcal{M}_3\left(\mathbf{t}, t_3\right) \left(\frac{\partial}{\partial x_3} \frac{x_3 - t_3}{\left(\left|\mathbf{x} - \mathbf{t}\right|^2 + \left(x_3 - t_3\right)^2\right)^{3/2}} \right) \Bigg|_{x_3 = h} \right] \mathrm{d}^3 t.$$

Taking Fourier transform of (3.5) in both x_1 and x_2 variables, we use (3.2), (3.3) and employ (3.4) twice: with $H := h - t_3$ in the first line of (3.5), and with $H := x_3 - t_3$ in the second one. We thus arrive at

$$(3.6) \qquad \hat{B}_3\left(\mathbf{k}, h\right) = \pi \int_{Q_3} e^{-2\pi(h - t_3)|\mathbf{k}|} \left(i k_1 \hat{\mathcal{M}}_1\left(\mathbf{k}, t_3\right) + i k_2 \hat{\mathcal{M}}_2\left(\mathbf{k}, t_3\right) + |\mathbf{k}| \hat{\mathcal{M}}_3\left(\mathbf{k}, t_3\right) \right) \mathrm{d} t_3,$$

where Q_3 denotes the vertical projection of the set Q.

We note that even though (2.2)–(2.3), (2.5), (2.7), (2.9) give estimates for tangential net moment components m_1, m_2, we shall restrict ourselves to dealing only with m_1. The situation with m_2 is completely analogous.

First, we are going to illustrate our strategy of the derivation of asymptotic estimates of the net magnetisation moment. Here, we shall be only concerned with the low-order formulas of Theorem 2.1 and we shall omit a rigorous justification step. Then, we shall proceed with formal justification and extension of the result to higher orders.

3.2. **Illustration of the basic idea of the derivation of the net moment estimates.** We are going to focus on deriving (2.2).

We take $k_2 = 0$ in expression (3.6) to obtain

$$(3.7) \qquad \hat{B}_3(k_1, 0, h) = \pi \int_{Q_3} e^{-2\pi(h-t_3)|k_1|} \left(i k_1 \hat{\mathcal{M}}_1(k_1, 0, t_3) + |k_1| \hat{\mathcal{M}}_3(k_1, 0, t_3) \right) dt_3.$$

Since the magnetisation distribution $\vec{\mathcal{M}}$ is compactly supported, the Fourier transforms $\hat{\mathcal{M}}_j(\mathbf{k}, t_3)$, $j \in \{1, 2, 3\}$, are entire functions in $k_1, k_2 \in \mathbb{C}$ for each $t_3 \in Q_3$, according to the Paley-Wiener theory (see, e.g., [24, Thm 4.1]). In particular, power-series expansion of $\hat{\mathcal{M}}_j(k_1, 0, t_3)$, $j \in \{1, 2, 3\}$, about the origin $k_1 = 0$ of the complex plane (Re k_1, Im k_1) gives

$$\hat{\mathcal{M}}_j(k_1, 0, t_3) = \hat{\mathcal{M}}_j(\mathbf{0}, t_3) + \partial_{k_1} \hat{\mathcal{M}}_j(\mathbf{0}, t_3) k_1 + \frac{1}{2} \partial_{k_1}^2 \hat{\mathcal{M}}_j(\mathbf{0}, t_3) k_1^2 + \mathcal{O}\left(|k_1|^3\right), \quad j \in \{1, 2, 3\}.$$

Combining this expansion with the straightforward identities

$$m_j = \int_{Q_3} \hat{\mathcal{M}}_j(\mathbf{0}, t_3) \, dt_3, \quad j \in \{1, 2, 3\},$$

$$2\pi i \langle M_j x_1 \rangle = \int_{Q_3} \partial_{k_1} \hat{\mathcal{M}}_j(\mathbf{0}, t_3) \, dt_3, \quad j \in \{1, 2, 3\},$$

and the Taylor expansion in $|k_1|$ of the exponential factor in (3.7)

$$e^{-2\pi(h-t_3)|k_1|} = 1 - 2\pi(h-t_3)|k_1| + 2\pi^2(h-t_3)^2 |k_1|^2 + \mathcal{O}\left(|k_1|^3\right),$$

we obtain

$$\hat{B}_3(k_1, 0, h) = \mathrm{Re}\, \hat{B}_3(k_1, 0, h) + i \,\mathrm{Im}\, \hat{B}_3(k_1, 0, h), \quad k_1 \in \mathbb{R},$$

$$\mathrm{Re}\, \hat{B}_3(k_1, 0, h) = \pi m_3 |k_1| - 2\pi^2 \left(\langle x_1 M_1 \rangle + \langle (h-x_3) M_3 \rangle \right) |k_1|^2$$

$$(3.8) \qquad\qquad + 2\pi^3 \left(2 \langle (h-x_3) x_1 M_1 \rangle + \left\langle (h-x_3)^2 M_3 \right\rangle - \langle x_1^2 M_3 \rangle \right) |k_1|^3 + \mathcal{O}\left(|k_1|^4\right),$$

$$\mathrm{Im}\, \hat{B}_3(k_1, 0, h) = \pi m_1 k_1 - 2\pi^2 \left(\langle (h-x_3) M_1 \rangle - \langle x_1 M_3 \rangle \right) k_1 |k_1|$$

$$(3.9) \qquad\qquad - 2\pi^3 \left(\langle x_1^2 M_1 \rangle - \left\langle (h-x_3)^2 M_1 \right\rangle + 2 \langle (h-x_3) x_1 M_3 \rangle \right) k_1^3 + \mathcal{O}\left(|k_1|^4\right).$$

Here, the remainder terms are uniformly small for all $t_3 \in Q_3$ due to the boundedness of the set Q_3.

On the other hand, we can write

$$(3.10) \qquad \hat{B}_3(k_1, 0, h) = \iint_{D_A} e^{2\pi i k_1 x_1} B_3(\mathbf{x}, h) \, d^2 x + \iint_{\mathbb{R}^2 \setminus D_A} e^{2\pi i k_1 x_1} B_3(\mathbf{x}, h) \, d^2 x.$$

We note that in the first term on the right-hand side of (3.10), the integration range is finite and hence an expansion in powers of k_1 simply follows from that of the exponential factor:

(3.11)

$$\iint_{D_A} e^{2\pi i k_1 x_1} B_3(\mathbf{x}, h) \, d^2 x = \iint_{D_A} B_3(\mathbf{x}, h) \, d^2 x + 2\pi i k_1 \iint_{D_A} x_1 B_3(\mathbf{x}, h) \, d^2 x - 2\pi^2 k_1^2 \iint_{D_A} x_1^2 B_3(\mathbf{x}, h) \, d^2 x$$

$$- \frac{4\pi^3}{3} i k_1^3 \iint_{D_A} x_1^3 B_3(\mathbf{x}, h) \, d^2 x + \mathcal{O}\left(|k_1|^4\right),$$

and hence

$$(3.12) \qquad \iint_{D_A} \cos(2\pi k_1 x_1) B_3(\mathbf{x}, h) \, d^2 x = \iint_{D_A} B_3(\mathbf{x}, h) \, d^2 x - 2\pi^2 k_1^2 \iint_{D_A} x_1^2 B_3(\mathbf{x}, h) \, d^2 x + \mathcal{O}\left(|k_1|^4\right),$$

$$(3.13) \qquad \iint_{D_A} \sin(2\pi k_1 x_1) B_3(\mathbf{x}, h) \, d^2 x = 2\pi k_1 \iint_{D_A} x_1 B_3(\mathbf{x}, h) \, d^2 x - \frac{4\pi^3}{3} k_1^3 \iint_{D_A} x_1^3 B_3(\mathbf{x}, h) \, d^2 x + \mathcal{O}\left(|k_1|^5\right).$$

Producing an expansion in powers of k_1 of the second term in (3.10) is much less straightforward and requires a preliminary simplification. More precisely, we expand the field $B_3(\mathbf{x}, h)$ for large $|\mathbf{x}|$ and, assuming largeness of the region D_A, retain only first few terms of this expansion. Namely, from (1.2), we have

$$(3.14) \qquad B_3(\mathbf{x}, h) = -\frac{m_3}{4\pi |\mathbf{x}|^3} + \frac{3}{4\pi} \frac{\left(\langle (h-x_3) M_1 \rangle - \langle x_1 M_3 \rangle \right) x_1 + \left(\langle (h-x_3) M_2 \rangle - \langle x_2 M_3 \rangle \right) x_2}{|\mathbf{x}|^5} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^5}\right).$$

Consequently, passing to the polar coordinates using $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $\mathrm{d}^2 x = r\mathrm{d}r\mathrm{d}\theta$, we can write

$$
\iint_{\mathbb{R}^2 \setminus D_A} e^{2\pi i k_1 x_1} B_3(\mathbf{x}, h) \, \mathrm{d}^2 x = -\frac{m_3}{4\pi} \int_A^\infty \int_0^{2\pi} e^{2\pi i k_1 r \cos\theta} \mathrm{d}\theta \frac{\mathrm{d}r}{r^2}
$$
$$
+ \frac{3}{4\pi} \left(\langle (h - x_3) M_1 \rangle - \langle x_1 M_3 \rangle \right) \int_A^\infty \int_0^{2\pi} e^{2\pi i k_1 r \cos\theta} \cos\theta \mathrm{d}\theta \frac{\mathrm{d}r}{r^3}
$$
$$
+ \frac{3}{4\pi} \left(\langle (h - x_3) M_2 \rangle - \langle x_2 M_3 \rangle \right) \int_A^\infty \int_0^{2\pi} e^{2\pi i k_1 r \cos\theta} \sin\theta \mathrm{d}\theta \frac{\mathrm{d}r}{r^3}
$$
$$
+ \mathcal{R}_{A,k_1},
$$

where the residue term $\mathcal{R}_{A,k_1}$ is expected to be $\mathcal{O}\left(1/A^3\right)$ for sufficiently small values of $|k_1|$.

Furthermore, taking real and imaginary parts of both sides, we obtain, respectively,

$$
\tag{3.15} \iint_{\mathbb{R}^2 \setminus D_A} \cos(2\pi k_1 x_1) B_3(\mathbf{x}, h) \, \mathrm{d}^2 x = -\frac{m_3}{4\pi} \int_A^\infty \int_0^{2\pi} \cos(2\pi k_1 r \cos\theta) \, \mathrm{d}\theta \frac{\mathrm{d}r}{r^2} + \operatorname{Re} \mathcal{R}_{A,k_1},
$$

$$
\tag{3.16}
$$
$$
\iint_{\mathbb{R}^2 \setminus D_A} \sin(2\pi k_1 x_1) B_3(\mathbf{x}, h) \, \mathrm{d}^2 x = \frac{3}{4\pi} \left(\langle (h - x_3) M_1 \rangle - \langle x_1 M_3 \rangle \right) \int_A^\infty \int_0^{2\pi} \sin(2\pi k_1 r \cos\theta) \cos\theta \mathrm{d}\theta \frac{\mathrm{d}r}{r^3} + \operatorname{Im} \mathcal{R}_{A,k_1},
$$

where we took into account that

$$
\int_0^{2\pi} \cos(2\pi k_1 r \cos\theta) \sin\theta \mathrm{d}\theta = \int_0^{2\pi} \cos(2\pi k_1 r \cos\theta) \cos\theta \mathrm{d}\theta = 0,
$$

$$
\int_0^{2\pi} \sin(2\pi k_1 r \cos\theta) \, \mathrm{d}\theta = \int_0^{2\pi} \sin(2\pi k_1 r \cos\theta) \sin\theta \mathrm{d}\theta = 0,
$$

according to the results of Lemma A.4.

Next, as it turns out (see Subsection 3.3 for more details), the integrals on the right-hand sides of (3.15)–(3.16) can be evaluated explicitly in terms of some cylindrical functions. Known representations of these special functions lead to the desired asymptotic expansions in powers of k_1. In particular, for small $|k_1|$, we can deduce that

$$
\tag{3.17} \int_A^\infty \int_0^{2\pi} \cos(2\pi k_1 r \cos\theta) \, \mathrm{d}\theta \frac{\mathrm{d}r}{r^2} = 2\pi |k_1| \int_{2\pi |k_1| A}^\infty \int_0^{2\pi} \cos(r\cos\theta) \, \mathrm{d}\theta \frac{\mathrm{d}r}{r^2}
$$
$$
= \frac{2\pi}{A} - 4\pi^2 |k_1| + 2\pi^3 A |k_1|^2 + \mathcal{O}\left(A^2 |k_1|^3\right),
$$

$$
\tag{3.18} \int_A^\infty \int_0^{2\pi} \sin(2\pi k_1 r \cos\theta) \cos\theta \mathrm{d}\theta \frac{\mathrm{d}r}{r^3} = 4\pi^2 k_1 |k_1| \int_{2\pi |k_1| A}^\infty \int_0^{2\pi} \sin(r\cos\theta) \cos\theta \mathrm{d}\theta \frac{\mathrm{d}r}{r^3}
$$
$$
= \frac{2\pi^2}{A} k_1 - \frac{8\pi^3}{3} k_1 |k_1| + \pi^4 A k_1^3 + \mathcal{O}\left(A^3 |k_1|^5\right),
$$

where the notation $\mathcal{O}\left(A^2 |k_1|^3\right)$ also hides the terms of powers of $|k_1|$ higher than 3 regardless of the presence of the A factors such as $\mathcal{O}\left(A^3 |k_1|^4\right)$.

Therefore, by taking the real part of (3.10) and using (3.12), (3.15) and (3.17), we obtain

$$
\operatorname{Re} \hat{B}_3(k_1, 0, h) = \iint_{D_A} B_3(\mathbf{x}, h) \, \mathrm{d}^2 x - 2\pi^2 k_1^2 \iint_{D_A} x_1^2 B_3(\mathbf{x}, h) \, \mathrm{d}^2 x
$$
$$
- \frac{m_3}{4\pi} \left(\frac{2\pi}{A} - 4\pi^2 |k_1| + 2\pi^3 A |k_1|^2 \right) + \operatorname{Re} \mathcal{R}_{A,k_1} + \mathcal{O}\left(A^2 |k_1|^3\right).
$$

Comparing this with (3.8) and, in particular, evaluating both expressions at $k_1 = 0$, we arrive at the following identity:

$$
\tag{3.19} \iint_{D_A} B_3(\mathbf{x}, h) \, \mathrm{d}^2 x - \frac{m_3}{2A} + \mathcal{O}\left(\frac{1}{A^3}\right) = 0,
$$

where we took into account that $\operatorname{Re} \mathcal{R}_{A,k_1}|_{k_1=0} = \mathcal{O}\left(1/A^3\right)$.

Similarly, taking the imaginary part of (3.10), we combine (3.13), (3.16) and (3.18) to deduce that

$$\text{Im } \hat{B}_3 (k_1, 0, h) = 2\pi k_1 \iint_{D_A} x_1 B_3 (\mathbf{x}, h) \, \mathrm{d}^2 x - \frac{4\pi^3}{3} k_1^3 \iint_{D_A} x_1^3 B_3 (\mathbf{x}, h) \, \mathrm{d}^2 x$$
$$+ \frac{3}{4\pi} \left(\langle (h - x_3) M_1 \rangle - \langle x_1 M_3 \rangle \right) \left(\frac{2\pi^2}{A} k_1 - \frac{8\pi^3}{3} k_1 |k_1| + \pi^4 A k_1^3 \right) + \text{Im } \mathcal{R}_{A, k_1} + \mathcal{O}\left(A^3 |k_1|^5 \right).$$

Comparison of this expression with (3.9) and matching the coefficients of the k_1 terms yields

$$(3.20) \qquad 2\pi \iint_{D_A} x_1 B_3 (\mathbf{x}, h) \, \mathrm{d}^2 x + \frac{3\pi}{2A} \left(\langle (h - x_3) M_1 \rangle - \langle x_1 M_3 \rangle \right) + \mathcal{O}\left(\frac{1}{A^3} \right) = \pi m_1,$$

where we assumed that, for sufficiently small $|k_1|$, we have $\text{Im } \mathcal{R}_{A, k_1} = \mathcal{O}\left(1/A^3 \right)$.

While (3.19), (3.20) imply estimates (2.4), (2.2), respectively, the derivation given above was not rigorous and required additional assumptions on the residue term $\mathcal{R}_{A, k_1}$ which was reasonably deemed to be sufficiently small for large A but was not estimated uniformly in k_1. We shall now proceed with rigorous analysis which will also make it possible to derive higher-order analogs of estimates (2.2), (2.4).

3.3. Rigorous analysis and higher-order asymptotic estimates.

Let us start by improving estimate (3.14). To this effect, we use the following elementary Taylor expansions, convergent for $|z| < 1$,

$$\frac{1}{(1 + z)^{3/2}} = 1 - \frac{3}{2} z + \frac{15}{8} z^2 - \frac{35}{16} z^3 + \mathcal{O}\left(z^4 \right),$$

$$\frac{1}{(1 + z)^{5/2}} = 1 - \frac{5}{2} z + \frac{35}{8} z^2 + \mathcal{O}\left(z^3 \right),$$

to obtain, for $t_1, t_2, t_3, h \in \mathbb{R}$, $t_3 \neq h$,

(3.21)

$$\frac{1}{\left[(x_1 - t_1)^2 + (x_2 - t_2)^2 + (h - t_3)^2 \right]^{3/2}} = \frac{1}{(x_1^2 + x_2^2)^{3/2}} \left[1 - 2 \frac{x_1 t_1 + x_2 t_2}{x_1^2 + x_2^2} + \frac{t_1^2 + t_2^2 + (h - t_3)^2}{x_1^2 + x_2^2} \right]^{-3/2}$$

$$= \frac{1}{(x_1^2 + x_2^2)^{3/2}} \left[1 + 3 \frac{x_1 t_1 + x_2 t_2}{x_1^2 + x_2^2} - \frac{3}{2} \frac{t_1^2 + t_2^2 + (h - t_3)^2}{x_1^2 + x_2^2} + \frac{15}{2} \frac{(x_1 t_1 + x_2 t_2)^2}{(x_1^2 + x_2^2)^2} \right.$$

$$\left. - \frac{15}{2} \frac{(x_1 t_1 + x_2 t_2) \left(t_1^2 + t_2^2 + (h - t_3)^2 \right)}{(x_1^2 + x_2^2)^2} + \frac{35}{2} \frac{(x_1 t_1 + x_2 t_2)^3}{(x_1^2 + x_2^2)^3} \right] + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^7} \right),$$

$$(3.22) \qquad \frac{1}{\left[(x_1 - t_1)^2 + (x_2 - t_2)^2 + (h - t_3)^2 \right]^{5/2}} = \frac{1}{(x_1^2 + x_2^2)^{5/2}} \left[1 + 5 \frac{x_1 t_1 + x_2 t_2}{x_1^2 + x_2^2} - \frac{5}{2} \frac{t_1^2 + t_2^2 + (h - t_3)^2}{x_1^2 + x_2^2} \right.$$

$$\left. + \frac{35}{2} \frac{(x_1 t_1 + x_2 t_2)^2}{(x_1^2 + x_2^2)^2} \right] + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^8} \right),$$

where $x_1, x_2 \in \mathbb{R}$, $|x_1| + |x_2| > 0$, are of a sufficiently large magnitude so that

$$(3.23) \qquad \left| \frac{t_1^2 + t_2^2 + (h - t_3)^2}{x_1^2 + x_2^2} - 2 \frac{x_1 t_1 + x_2 t_2}{x_1^2 + x_2^2} \right| < 1.$$

Expansions (3.21)–(3.22) imply that, for $|\mathbf{x}| \gg 1$, (1.2) can be written as

$$(3.24) \qquad B_3 (\mathbf{x}, h) = B_3^{\text{asympt}} (\mathbf{x}, h) + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^7} \right),$$

with

$$(3.25) \qquad B_3^{\text{asympt}} (\mathbf{x}, h) := \frac{c_{0,0}}{|\mathbf{x}|^3} + \frac{c_{1,0} x_1 + c_{0,1} x_2}{|\mathbf{x}|^5} + \frac{c_{2,0} x_1^2 + c_{0,2} x_2^2 + c_{1,1} x_1 x_2}{|\mathbf{x}|^7}$$

$$+ \frac{c_{3,0} x_1^3 + c_{0,3} x_2^3 + c_{2,1} x_1^2 x_2 + c_{1,2} x_1 x_2^2}{|\mathbf{x}|^9},$$

$$(3.26) \qquad c_{0,0} := -\frac{m_3}{4\pi},$$

$$(3.27) \qquad c_{1,0} := \frac{3}{4\pi} \left[\langle (h - x_3) M_1 \rangle - \langle x_1 M_3 \rangle \right], \qquad c_{0,1} := \frac{3}{4\pi} \left[\langle (h - x_3) M_2 \rangle - \langle x_2 M_3 \rangle \right],$$

$$(3.28) \qquad \begin{aligned} c_{2,0} := & \frac{3}{8\pi} \Big[8 \langle (h - x_3) x_1 M_1 \rangle - 4 \langle x_1^2 M_3 \rangle + \langle x_2^2 M_3 \rangle \\ & -2 \langle (h - x_3) x_2 M_2 \rangle + 3 \left\langle (h - x_3)^2 M_3 \right\rangle \Big], \end{aligned}$$

$$(3.29) \qquad \begin{aligned} c_{0,2} := & \frac{3}{8\pi} \Big[8 \langle (h - x_3) x_2 M_2 \rangle - 4 \langle x_2^2 M_3 \rangle + \langle x_1^2 M_3 \rangle \\ & -2 \langle (h - x_3) x_1 M_1 \rangle + 3 \left\langle (h - x_3)^2 M_3 \right\rangle \Big], \end{aligned}$$

$$(3.30) \qquad c_{1,1} := \frac{15}{4\pi} \left[\langle (h - x_3) x_2 M_1 \rangle + \langle (h - x_3) x_1 M_2 \rangle - \langle x_1 x_2 M_3 \rangle \right],$$

$$(3.31) \qquad \begin{aligned} c_{3,0} := & \frac{5}{8\pi} \Big[12 \langle (h - x_3) x_1^2 M_1 \rangle - 4 \langle x_1^3 M_3 \rangle - 3 \langle (h - x_3) x_2^2 M_1 \rangle - 3 \left\langle (h - x_3)^3 M_1 \right\rangle \\ & -6 \langle (h - x_3) x_1 x_2 M_2 \rangle + 3 \langle x_1 x_2^2 M_3 \rangle + 9 \left\langle (h - x_3)^2 x_1 M_3 \right\rangle \Big], \end{aligned}$$

$$(3.32) \qquad \begin{aligned} c_{0,3} := & \frac{5}{8\pi} \Big[12 \langle (h - x_3) x_2^2 M_2 \rangle - 4 \langle x_2^3 M_3 \rangle - 3 \langle (h - x_3) x_1^2 M_2 \rangle - 3 \left\langle (h - x_3)^3 M_2 \right\rangle \\ & -6 \langle (h - x_3) x_1 x_2 M_1 \rangle + 3 \langle x_1^2 x_2 M_3 \rangle + 9 \left\langle (h - x_3)^2 x_2 M_3 \right\rangle \Big], \end{aligned}$$

$$(3.33) \qquad \begin{aligned} c_{2,1} := & \frac{15}{8\pi} \Big[6 \langle (h - x_3) x_1^2 M_2 \rangle + 12 \langle (h - x_3) x_1 x_2 M_1 \rangle - 6 \langle x_1^2 x_2 M_3 \rangle \\ & -3 \langle (h - x_3) x_2^2 M_2 \rangle - \left\langle (h - x_3)^3 M_2 \right\rangle + \langle x_2^3 M_3 \rangle + 3 \left\langle (h - x_3)^2 x_2 M_3 \right\rangle \Big], \end{aligned}$$

$$(3.34) \qquad \begin{aligned} c_{1,2} := & \frac{15}{8\pi} \Big[6 \langle (h - x_3) x_2^2 M_1 \rangle + 12 \langle (h - x_3) x_1 x_2 M_2 \rangle - 6 \langle x_1 x_2^2 M_3 \rangle \\ & -3 \langle (h - x_3) x_1^2 M_1 \rangle - \left\langle (h - x_3)^3 M_1 \right\rangle + \langle x_1^3 M_3 \rangle + 3 \left\langle (h - x_3)^2 x_1 M_3 \right\rangle \Big], \end{aligned}$$

and condition (3.23), for our particular context, rewrites as (2.1).

We are going to pursue the idea outlined in the previous subsection. Namely, comparing a series expansion of (3.7) about $k_1 = 0$ with that of (3.10), we shall deduce a set of identities which relate magnetisation moments to the integrals of the measured data $B_3(\mathbf{x}, h)$ on D_A. More precisely, using compactness of the support of the magnetisation $\vec{\mathcal{M}}$, it follows from (3.7) that $\mathrm{Re}\ \hat{B}_3(k_1, 0, h)$ is a convergent series in powers of $|k_1|$ whereas $\mathrm{Im}\ \hat{B}_3(k_1, 0, h)$ is a power series in $|k_1|$ multiplied by k_1. To facilitate the situation of matching the coefficients of different representations, we shall focus on the region $k_1 > 0$. Consequently, in what follows, evaluation at $k_1 = 0^+$ will mean the limiting value at $k_1 = 0$ taken from the positive semiaxis ($k_1 > 0$).

3.3.1. *Tangential components of the net moment.* Expanding the integrand in (3.7) in power series in a positive neighborhood of $k_1 = 0$ and taking the imaginary part, we deduce:

$$(3.35) \qquad \partial_{k_1} \left[\mathrm{Im}\ \hat{B}_3(k_1, 0, h) \right]\Big|_{k_1 = 0^+} = \pi m_1 =: d_1,$$

$$(3.36) \qquad \begin{aligned} \frac{1}{6} \partial_{k_1}^3 \left[\mathrm{Im}\ \hat{B}_3(k_1, 0, h) \right]\Big|_{k_1 = 0^+} &= 2\pi^3 \left[\left\langle (h - x_3)^2 M_1 \right\rangle - \langle x_1^2 M_1 \rangle - 2 \langle (h - x_3) x_1 M_3 \rangle \right] \\ &=: d_3, \end{aligned}$$

$$(3.37) \qquad \frac{1}{5!} \, \partial_{k_1}^5 \left[\mathrm{Im} \, \hat{B}_3 \left(k_1, 0, h \right) \right] \Big|_{k_1=0^+} = \frac{2\pi^5}{3} \Big[\left\langle (h-x_3)^4 \, M_1 \right\rangle - 6 \left\langle (h-x_3)^2 \, x_1^2 M_1 \right\rangle + \left\langle x_1^4 M_1 \right\rangle$$

$$-4 \left\langle (h-x_3)^3 \, x_1 M_3 \right\rangle + 4 \left\langle (h-x_3) \, x_1^3 M_3 \right\rangle \Big]$$

$$=: d_5,$$

(3.38)

$$\frac{1}{7!} \, \partial_{k_1}^7 \left[\mathrm{Im} \, \hat{B}_3 \left(k_1, 0, h \right) \right] \Big|_{k_1=0^+} = \frac{4\pi^7}{45} \Big[\left\langle (h-x_3)^6 \, M_1 \right\rangle - 15 \left\langle (h-x_3)^4 \, x_1^2 M_1 \right\rangle + 15 \left\langle (h-x_3)^2 \, x_1^4 M_1 \right\rangle - \left\langle x_1^6 M_1 \right\rangle$$

$$-6 \left\langle (h-x_3)^5 \, x_1 M_3 \right\rangle + 20 \left\langle (h-x_3)^3 \, x_1^3 M_3 \right\rangle - 6 \left\langle (h-x_3) \, x_1^5 M_3 \right\rangle \Big]$$

$$=: d_7,$$

$$(3.39) \qquad \frac{1}{9!} \, \partial_{k_1}^9 \left[\mathrm{Im} \, \hat{B}_3 \left(k_1, 0, h \right) \right] \Big|_{k_1=0^+} = \frac{2\pi^9}{315} \Big[\left\langle (h-x_3)^8 \, M_1 \right\rangle - 28 \left\langle (h-x_3)^6 \, x_1^2 M_1 \right\rangle + 70 \left\langle (h-x_3)^4 \, x_1^4 M_1 \right\rangle$$

$$- 28 \left\langle (h-x_3)^2 \, x_1^6 M_1 \right\rangle + \left\langle x_1^8 M_1 \right\rangle - 8 \left\langle (h-x_3)^7 \, x_1 M_3 \right\rangle$$

$$+56 \left\langle (h-x_3)^5 \, x_1^3 M_3 \right\rangle - 56 \left\langle (h-x_3)^3 \, x_1^5 M_3 \right\rangle + 8 \left\langle (h-x_3) \, x_1^7 M_3 \right\rangle \Big]$$

$$=: d_9,$$

(3.40)

$$\frac{1}{11!} \, \partial_{k_1}^{11} \left[\mathrm{Im} \, \hat{B}_3 \left(k_1, 0, h \right) \right] \Big|_{k_1=0^+} = \frac{4\pi^{11}}{14175} \Big[\left\langle (h-x_3)^{10} \, M_1 \right\rangle - 45 \left\langle (h-x_3)^8 \, x_1^2 M_1 \right\rangle + 210 \left\langle (h-x_3)^6 \, x_1^4 M_1 \right\rangle$$

$$- 210 \left\langle (h-x_3)^4 \, x_1^6 M_1 \right\rangle + 45 \left\langle (h-x_3)^2 \, x_1^8 M_1 \right\rangle - \left\langle x_1^{10} M_1 \right\rangle - 10 \left\langle (h-x_3)^9 \, x_1 M_3 \right\rangle$$

$$+ 120 \left\langle (h-x_3)^7 \, x_1^3 M_3 \right\rangle - 252 \left\langle (h-x_3)^5 \, x_1^5 M_3 \right\rangle + 120 \left\langle (h-x_3)^3 \, x_1^7 M_3 \right\rangle$$

$$-10 \left\langle (h-x_3) \, x_1^9 M_3 \right\rangle \Big]$$

$$=: d_{11}.$$

On the other hand, from (3.10), we have, for $n \in \mathbb{N}_0$,

$$(3.41) \quad \partial_{k_1}^{2n+1} \left[\mathrm{Im} \, \hat{B}_3 \left(k_1, 0, h \right) \right] \Big|_{k_1=0^+} = (-1)^n \, (2\pi)^{2n+1} \iint_{D_A} x_1^{2n+1} B_3 \left(\mathbf{x}, h \right) \mathrm{d}^2 x$$

$$+ \partial_{k_1}^{2n+1} \left(\iint_{\mathbb{R}^2 \backslash D_A} \sin \left(2\pi k_1 x_1 \right) B_3^{\mathrm{asympt}} \left(\mathbf{x}, h \right) \mathrm{d}^2 x \right) \Bigg|_{k_1=0^+}$$

$$+ \partial_{k_1}^{2n+1} \left(\iint_{\mathbb{R}^2 \backslash D_A} \sin \left(2\pi k_1 x_1 \right) \left[B_3 \left(\mathbf{x}, h \right) - B_3^{\mathrm{asympt}} \left(\mathbf{x}, h \right) \right] \mathrm{d}^2 x \right) \Bigg|_{k_1=0^+}.$$

We shall proceed in 3 steps. First, we evaluate the integral

$$(3.42) \qquad \mathcal{I}^{\mathrm{sin}} \equiv \mathcal{I}^{\mathrm{sin}} \left(k_1, A \right) := \iint_{\mathbb{R}^2 \backslash D_A} \sin \left(2\pi k_1 x_1 \right) B_3^{\mathrm{asympt}} \left(\mathbf{x}, h \right) \mathrm{d}^2 x,$$

and compute its derivatives appearing on the second line of (3.41), hence obtaining a set of valuable identities. Second, we estimate the derivatives of the remainder

$$(3.43) \quad \mathcal{R}_{2n+1}^{\mathrm{sin}} \equiv \mathcal{R}_{2n+1}^{\mathrm{sin}} \left(A \right) := \partial_{k_1}^{2n+1} \left(\iint_{\mathbb{R}^2 \backslash D_A} \sin \left(2\pi k_1 x_1 \right) \left[B_3 \left(\mathbf{x}, h \right) - B_3^{\mathrm{asympt}} \left(\mathbf{x}, h \right) \right] \mathrm{d}^2 x \right) \Bigg|_{k_1=0^+}, \quad n \in \mathbb{N}_0,$$

in order to show that the contribution of the term in the third line of (3.41) is not significant for large A (for the chosen order of the asymptotic expansion). Finally, at the last step, we combine the obtained identities and derive asymptotic formulas (2.2)–(2.3), (2.5), (2.7), (2.9) in a rigorously justified fashion.

Step 1: Derivation of the set of identities. Using (3.25) and passing to the polar coordinates using $x = r\cos\theta$, $y = r\sin\theta$, $\mathrm{d}^2 x = r\mathrm{d}r\mathrm{d}\theta$, we obtain from (3.42)

$$(3.44) \qquad \mathcal{I}^{\sin} = c_{1,0}\mathcal{I}_1^{\sin} + + c_{3,0}\mathcal{I}_2^{\sin} + c_{1,2}\mathcal{I}_3^{\sin},$$

where

$$(3.45) \qquad \mathcal{I}_1^{\sin} := \int_A^\infty \int_0^{2\pi} \sin(2\pi k_1 r\cos\theta)\cos\theta\mathrm{d}\theta\frac{\mathrm{d}r}{r^3},$$

$$(3.46) \qquad \mathcal{I}_2^{\sin} := \int_A^\infty \int_0^{2\pi} \sin(2\pi k_1 r\cos\theta)\cos^3\theta\mathrm{d}\theta\frac{\mathrm{d}r}{r^5},$$

$$(3.47) \qquad \mathcal{I}_3^{\sin} := \int_A^\infty \int_0^{2\pi} \sin(2\pi k_1 r\cos\theta)\cos\theta\sin^2\theta\mathrm{d}\theta\frac{\mathrm{d}r}{r^5}.$$

Here, we used results (A.42)–(A.43) of Lemma A.4 multiple times to deduce vanishing of the integrals associated with the terms which involve $c_{0,0}$, $c_{0,1}$, $c_{2,0}$, $c_{0,2}$, $c_{1,1}$, $c_{0,3}$, $c_{2,1}$.

We now employ the integral representation of Bessel functions, given in (A.5), to rewrite, for $k_1 > 0$,

$$(3.48) \qquad \mathcal{I}_1^{\sin} = 2\pi \int_A^\infty J_1(2\pi k_1 r)\frac{\mathrm{d}r}{r^3} = (2\pi)^3 k_1^2 \int_{2\pi k_1 A}^\infty \frac{J_1(x)}{x^3}\mathrm{d}x,$$

$$(3.49) \qquad \mathcal{I}_2^{\sin} = -2\pi \int_A^\infty J_1''(2\pi k_1 r)\frac{\mathrm{d}r}{r^5} = -(2\pi)^5 k_1^4 \int_{2\pi k_1 A}^\infty \frac{J_1''(x)}{x^5}\mathrm{d}x$$
$$= (2\pi)^5 k_1^4 \left(\frac{5J_1(\rho)}{\rho^6} + \frac{J_1'(\rho)}{\rho^5} - 30\int_\rho^\infty \frac{J_1(x)}{x^7}\mathrm{d}x\right)\Bigg|_{\rho=2\pi k_1 A},$$

$$(3.50) \qquad \mathcal{I}_3^{\sin} = 2\pi \int_A^\infty [J_1(2\pi k_1 r) + J_1''(2\pi k_1 r)]\frac{\mathrm{d}r}{r^5} = (2\pi)^5 k_1^4 \int_{2\pi k_1 A}^\infty \frac{J_1(x) + J_1''(x)}{x^5}\mathrm{d}x,$$
$$= -(2\pi)^5 k_1^4 \left(\frac{5J_1(\rho)}{\rho^6} + \frac{J_1'(\rho)}{\rho^5} - \int_\rho^\infty \frac{J_1(x)}{x^5}\mathrm{d}x - 30\int_\rho^\infty \frac{J_1(x)}{x^7}\mathrm{d}x\right)\Bigg|_{\rho=2\pi k_1 A}.$$

Note that, in (3.49)–(3.50), we employed integration by parts twice using the asymptotic behavior of J_1 given in (A.6).

Using the results of Lemmas A.2–A.3, we have

$$(3.51) \quad \mathcal{I}_1^{\sin} = \frac{(2\pi)^2 k_1}{A}\left(\rho\int_\rho^\infty \frac{J_1(x)}{x^3}\mathrm{d}x\right)\Bigg|_{\rho=2\pi k_1 A}$$
$$= \frac{(2\pi)^2 k_1}{3A}\left[J_0(\rho) + \frac{J_1(\rho)}{\rho} - \rho - \rho J_1(\rho) + \rho^2 J_0(\rho) - \frac{\pi}{2}\rho^2 J_0(\rho)H_1(\rho) + \frac{\pi}{2}\rho^2 J_1(\rho)H_0(\rho)\right]\Bigg|_{\rho=2\pi k_1 A},$$

$$(3.52) \qquad \mathcal{I}_2^{\sin} = \frac{(2\pi)^2 k_1}{A^3}\left(\frac{5J_1(\rho)}{\rho^3} + \frac{J_1'(\rho)}{\rho^2} - 30\rho^3\int_\rho^\infty \frac{J_1(x)}{x^7}\mathrm{d}x\right)\Bigg|_{\rho=2\pi k_1 A}$$
$$= \frac{(2\pi)^2 k_1}{105 A^3}\left[-\frac{15J_1(\rho)}{\rho^3} + \frac{15J_1'(\rho)}{\rho^2} + \frac{24J_1(\rho)}{\rho} + 6J_1'(\rho) - 2\rho^2 J_0(\rho) - 2\rho J_1(\rho)\right.$$
$$\left. + 2\rho^3 + 2\rho^3 J_1(\rho) - 2\rho^4 J_0(\rho) + \pi\rho^4 J_0(\rho)H_1(\rho) - \pi\rho^4 J_1(\rho)H_0(\rho)\right]\Bigg|_{\rho=2\pi k_1 A},$$

$$(3.53) \qquad \mathcal{I}_3^{\sin} = -\frac{(2\pi)^2 k_1}{A^3}\left(\frac{5J_1(\rho)}{\rho^3} + \frac{J_1'(\rho)}{\rho^2} - \rho^3\int_\rho^\infty \frac{J_1(x)}{x^5}\mathrm{d}x - 30\rho^3\int_\rho^\infty \frac{J_1(x)}{x^7}\mathrm{d}x\right)\Bigg|_{\rho=2\pi k_1 A}$$
$$= \frac{(2\pi)^2 k_1}{105 A^3}\left[\frac{15J_1(\rho)}{\rho^3} - \frac{15J_1'(\rho)}{\rho^2} + \frac{4J_1(\rho)}{\rho} + J_1'(\rho) - \frac{\rho^2 J_0(\rho)}{3} - \frac{\rho J_1(\rho)}{3}\right.$$
$$\left. + \frac{\rho^3}{3} + \frac{\rho^3 J_1(\rho)}{3} - \frac{\rho^4 J_0(\rho)}{3} + \frac{\pi\rho^4}{6}J_0(\rho)H_1(\rho) - \frac{\pi\rho^4}{6}J_1(\rho)H_0(\rho)\right]\Bigg|_{\rho=2\pi k_1 A}.$$

Therefore, (3.44) together with (3.51)–(3.53) furnishes an explicit form of (3.42). In particular, using (A.2), (A.11), we can compute

$$(3.54) \qquad \partial_{k_1} \mathcal{I}^{\mathrm{sin}}\big|_{k_1=0^+} = \frac{1}{2} (2\pi)^2 \left(\frac{c_{1,0}}{A} + \frac{3c_{3,0} + c_{1,2}}{12A^3} \right),$$

$$(3.55) \qquad \partial_{k_1}^3 \mathcal{I}^{\mathrm{sin}}\big|_{k_1=0^+} = \frac{3}{8} (2\pi)^4 A^2 \left(\frac{c_{1,0}}{A} - \frac{5c_{3,0} + c_{1,2}}{6A^3} \right),$$

$$(3.56) \qquad \partial_{k_1}^5 \mathcal{I}^{\mathrm{sin}}\big|_{k_1=0^+} = -\frac{5}{16} (2\pi)^6 A^4 \left(\frac{c_{1,0}}{3A} + \frac{7c_{3,0} + c_{1,2}}{8A^3} \right),$$

$$(3.57) \qquad \partial_{k_1}^7 \mathcal{I}^{\mathrm{sin}}\big|_{k_1=0^+} = \frac{7}{128} (2\pi)^8 A^6 \left(\frac{c_{1,0}}{A} + \frac{9c_{3,0} + c_{1,2}}{6A^3} \right),$$

$$(3.58) \qquad \partial_{k_1}^9 \mathcal{I}^{\mathrm{sin}}\big|_{k_1=0^+} = -\frac{21}{256} (2\pi)^{10} A^8 \left(\frac{3c_{1,0}}{7A} + \frac{11c_{3,0} + c_{1,2}}{20A^3} \right),$$

$$(3.59) \qquad \partial_{k_1}^{11} \mathcal{I}^{\mathrm{sin}}\big|_{k_1=0^+} = \frac{33}{14336} (2\pi)^{12} A^{10} \left(\frac{98c_{1,0}}{9A} + \frac{13c_{3,0} + c_{1,2}}{A^3} \right).$$

Taking into account (3.42), (3.43), we use (3.35)–(3.40) and (3.54)–(3.59) in (3.41) with $n = 0, \dots, 5$, respectively, and thus arrive at the following set of identities:

$$(3.60) \qquad 2\pi \iint_{D_A} x_1 B_3 (\mathbf{x}, h) \, \mathrm{d}^2 x + 2\pi^2 \left(\frac{c_{1,0}}{A} + \frac{3c_{3,0} + c_{1,2}}{12A^3} \right) + \mathcal{R}_1^{\mathrm{sin}} = d_1,$$

$$(3.61) \qquad -\frac{4\pi^3}{3} \iint_{D_A} x_1^3 B_3 (\mathbf{x}, h) \, \mathrm{d}^2 x + \pi^4 A^2 \left(\frac{c_{1,0}}{A} - \frac{5c_{3,0} + c_{1,2}}{6A^3} \right) + \frac{1}{6} \mathcal{R}_3^{\mathrm{sin}} = d_3,$$

$$(3.62) \qquad \frac{4\pi^5}{15} \iint_{D_A} x_1^5 B_3 (\mathbf{x}, h) \, \mathrm{d}^2 x - \frac{\pi^6 A^4}{6} \left(\frac{c_{1,0}}{3A} + \frac{7c_{3,0} + c_{1,2}}{8A^3} \right) + \frac{1}{5!} \mathcal{R}_5^{\mathrm{sin}} = d_5,$$

$$(3.63) \qquad -\frac{(2\pi)^7}{5040} \iint_{D_A} x_1^7 B_3 (\mathbf{x}, h) \, \mathrm{d}^2 x + \frac{(2\pi)^8 A^6}{92160} \left(\frac{c_{1,0}}{A} + \frac{9c_{3,0} + c_{1,2}}{6A^3} \right) + \frac{1}{7!} \mathcal{R}_7^{\mathrm{sin}} = d_7,$$

$$(3.64) \qquad \frac{(2\pi)^9}{362880} \iint_{D_A} x_1^9 B_3 (\mathbf{x}, h) \, \mathrm{d}^2 x - \frac{7 (2\pi)^{10} A^8}{10321920} \left(\frac{c_{1,0}}{7A} + \frac{11c_{3,0} + c_{1,2}}{60A^3} \right) + \frac{1}{9!} \mathcal{R}_9^{\mathrm{sin}} = d_9,$$

$$(3.65) \qquad -\frac{(2\pi)^{11}}{39916800} \iint_{D_A} x_1^{11} B_3 (\mathbf{x}, h) \, \mathrm{d}^2 x + \frac{(2\pi)^{12} A^{10}}{17340825600} \left(\frac{98c_{1,0}}{9A} + \frac{13c_{3,0} + c_{1,2}}{A^3} \right) + \frac{1}{11!} \mathcal{R}_{11}^{\mathrm{sin}} = d_{11}.$$

<u>Step 2: Analysis of the remainder terms $\mathcal{R}_{2n+1}^{\mathrm{sin}}$ for $0 \le n \le 5$.</u> We shall now show that the remainder terms $\mathcal{R}_{2n+1}^{\mathrm{sin}}$ with $n = 0, \dots, 5$ given by (3.43) (with $k_1 > 0$, as assumed before), can be estimated, for $A \gg 1$, as follows

$$(3.66) \qquad \mathcal{R}_{2n+1}^{\mathrm{sin}} = \mathcal{O}\left(\frac{1}{A^{5-2n}} \right), \qquad 0 \le n \le 5.$$

Proceeding with higher-order terms in the expansions in (3.21)–(3.22), and hence also in (3.24), we notice that we can write, for $N \ge 4$,

$$(3.67) \qquad B_3 (\mathbf{x}, h) - B_3^{\mathrm{asympt}} (\mathbf{x}, h) = \sum_{q=4}^{N} \sum_{\substack{l_1, l_2 \ge 0, \\ l_1 + l_2 = q}} c_{l_1, l_2} \frac{x_1^{l_1} x_2^{l_2}}{|\mathbf{x}|^{2q+3}} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{N+4}} \right) =: L_N (\mathbf{x}) + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{N+4}} \right),$$

with some constants $c_{l_1, l_2} \in \mathbb{R}$ for $l_1, l_2 \in \mathbb{N}_0$. Consequently, we consider, for $n = 0, \dots, 5$,

$$(3.68) \qquad \mathcal{R}_{2n+1}^{\mathrm{sin}} = \partial_{k_1}^{2n+1} \left(\iint_{\mathbb{R}^2 \setminus D_A} \sin (2\pi k_1 x_1) L_N (\mathbf{x}) \, \mathrm{d}^2 x \right)\Bigg|_{k_1=0^+}$$
$$+ \partial_{k_1}^{2n+1} \left(\iint_{\mathbb{R}^2 \setminus D_A} \sin (2\pi k_1 x_1) \left[B_3 (\mathbf{x}, h) - B_3^{\mathrm{asympt}} (\mathbf{x}, h) - L_N (\mathbf{x}) \right] \mathrm{d}^2 x \right)\Bigg|_{k_1=0^+}.$$

First of all, we deal with the term on the second line. The integrand is regular and, for $N > 9$, it decays at infinity sufficiently fast so that the differential operator $\partial_{k_1}^{2n+1}$ with $0 \leq n \leq 5$ can be passed under the integral sign. We can thus estimate

$$\left| \partial_{k_1}^{2n+1} \left(\iint_{\mathbb{R}^2 \setminus D_A} \sin\left(2\pi k_1 x_1\right) \left[B_3\left(\mathbf{x}, h\right) - B_3^{\mathrm{asympt}}\left(\mathbf{x}, h\right) - L_N\left(\mathbf{x}\right) \right] \mathrm{d}^2 x \right) \Big|_{k_1 = 0^+} \right|$$

$$= \left| \left(\iint_{\mathbb{R}^2 \setminus D_A} \frac{\left(2\pi x_1\right)^{2n+1}}{\left|\mathbf{x}\right|^{N+4}} \left|\mathbf{x}\right|^{N+4} \left[B_3\left(\mathbf{x}, h\right) - B_3^{\mathrm{asympt}}\left(\mathbf{x}, h\right) - L_N\left(\mathbf{x}\right) \right] \mathrm{d}^2 x \right) \right|$$

$$\leq \left(2\pi\right)^{2(n+1)} C_N \int_A^\infty \frac{\mathrm{d}r}{r^{N-2n+1}} = \frac{\left(2\pi\right)^{2(n+1)} C_N}{\left(N - 2n\right)} \frac{1}{A^{N-2n}}$$

for some constant $C_N > 0$ such that

$$\left|\mathbf{x}\right|^{N+4} \left| B_3\left(\mathbf{x}, h\right) - B_3^{\mathrm{asympt}}\left(\mathbf{x}, h\right) - L_N\left(\mathbf{x}\right) \right| \leq C_N, \quad \left|\mathbf{x}\right| \geq A,$$

and such a bound is possible due to the remainder estimate $\mathcal{O}\left(1/\left|\mathbf{x}\right|^{N+4}\right)$ in (3.67). Here, in the third line of the estimates, we used the fact that the integral in r converges for $N > 2n - 1$, $0 \leq n \leq 5$, which is true for $N > 9$. For such N, the obtained estimate of order $\mathcal{O}\left(1/A^{N-2n}\right)$ is clearly even better than was aimed for (recall (3.66)).

We now fix $N = 10$ and proceed with estimating the term in the first line of (3.68). Upon substitution of (3.67) in (3.43) and use of polar coordinates (with $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, as before), let us observe that, due to Lemma A.4 (namely, identities (A.42)–(A.43)), the only non-vanishing terms stemming from the L_N part are those proportional to

$$(3.69) \qquad \int_A^\infty \int_0^{2\pi} \sin\left(2\pi k_1 r \cos\theta\right) \cos^{2(p-l)+1}\theta \, \sin^{2l}\theta \mathrm{d}\theta \frac{\mathrm{d}r}{r^{2p+3}}, \qquad 0 \leq l \leq p, \quad p \geq 2.$$

Since we can write

$$\sin^{2l}\theta = \left(1 - \cos^2\theta\right)^l = \sum_{j=0}^l \binom{l}{j} (-1)^{l-j} \cos^{2(l-j)}\theta,$$

with $\binom{l}{j}$ denoting a binomial coefficient, we deduce that, to estimate $\mathcal{R}_{2n+1}^{\sin}$, it suffices only to consider the quantities

$$(3.70) \qquad \mathcal{S}_{p,j} := \int_A^\infty \int_0^{2\pi} \sin\left(2\pi k_1 r \cos\theta\right) \cos^{2j+1}\theta \, \mathrm{d}\theta \frac{\mathrm{d}r}{r^{2p+3}}, \qquad 0 \leq j \leq p, \quad p \geq 2,$$

and, in particular, their derivatives evaluated at $k_1 = 0$ from the right: $\partial_{k_1}^{2n+1} \mathcal{S}_{p,j}\big|_{k_1 = 0^+}$, $0 \leq n \leq 5$.

Note that the relation between p (3.69)–(3.70) and q in (3.67) is $q = 2p + 1$, $p \geq 2$. In other words, in the asymptotic expansion of the field B_3 at infinity, not every term contributes to $\mathcal{R}_{2n+1}^{\sin}$, but only the terms of every second order in $1/A$, i.e., $\mathcal{O}\left(1/A^8\right)$, $\mathcal{O}\left(1/A^{10}\right)$ and so on. Therefore, we can consider $\mathcal{S}_{p,j}$ only for $p \leq \lfloor \frac{N-1}{2} \rfloor = 8$, where $\lfloor x \rfloor$ designates the integer part of x.

To sum up, we need to show that, for all $0 \leq j \leq p$, $2 \leq p \leq 8$ and $0 \leq n \leq 5$, we are able to produce an estimate

$$(3.71) \qquad \partial_{k_1}^{2n+1} \mathcal{S}_{p,j}\big|_{k_1 = 0^+} = \mathcal{O}\left(\frac{1}{A^{5-2n}}\right).$$

For $0 \leq n \leq p$, we have

$$\partial_{k_1}^{2n+1} \mathcal{S}_{p,j}\big|_{k_1 = 0^+} = (-1)^n \left(2\pi\right)^{2n+1} \int_A^\infty \int_0^{2\pi} \cos^{2j+1}\theta \, \mathrm{d}\theta \frac{\mathrm{d}r}{r^{2(p-n)+2}},$$

and hence

$$\left| \partial_{k_1}^{2n+1} \mathcal{S}_{p,j}\big|_{k_1 = 0^+} \right| \leq \left(2\pi\right)^{2(n+1)} \int_A^\infty \frac{\mathrm{d}r}{r^{2(p-n)+2}} = \mathcal{O}\left(\frac{1}{A^{2(p-n)+1}}\right),$$

where the convergence of the last integral is due to $p \geq n$. The obtained estimate is in agreement with (3.71) since $p \geq 2$.

To treat the case $n \geq p + 1$, a more careful estimate is needed. To this end, it is convenient to make use of the integral representation of the Bessel function J_1 given by (A.5) and rewrite (3.70) as

$$\mathcal{S}_{p,j} = (-1)^j 2\pi \int_A^\infty J_1^{(2j)}\left(2\pi k_1 r\right) \frac{\mathrm{d}r}{r^{2p+3}}, \qquad 0 \leq j \leq p, \quad p \geq 2.$$

We then evaluate

$$(3.72) \qquad \partial_{k_1}^{2n+1} \mathcal{S}_{p,j} = (-1)^j (2\pi)^{2p+3} \partial_{k_1}^{2(n-p)-1} \int_A^\infty J_1^{(2j+2p+2)} (2\pi k_1 r) \frac{\mathrm{d}r}{r}$$

$$= (-1)^{j+1} (2\pi)^{2p+4} A \partial_{k_1}^{2(n-p)-2} \left. \frac{J_1^{(2j+2p+2)} (\rho)}{\rho} \right|_{\rho=2\pi k_1 A}$$

$$= (-1)^{j+1} (2\pi)^{2(n+1)} A^{2(n-p)-1} \left. \frac{\mathrm{d}^{2(n-p)-2}}{\mathrm{d}\rho^{2(n-p)-2}} \left(\frac{J_1^{(2j+2p+2)} (\rho)}{\rho} \right) \right|_{\rho=2\pi k_1 A},$$

where, in passing the differential operator $\partial_{k_1}^{2p+2}$ under the integral sign, we took into account the asymptotic behavior at infinity of J_1 given by (A.6) and, on the second line, employed the following identity valid for $k_1 > 0$:

$$\partial_{k_1} \int_A^\infty J_1^{(2j+2p+2)} (2\pi k_1 r) \frac{\mathrm{d}r}{r} = \partial_{k_1} \int_{2\pi k_1 A}^\infty J_1^{(2j+2p+2)} (\rho) \frac{\mathrm{d}\rho}{\rho}$$

$$= -2\pi A \left. \frac{J_1^{(2j+2p+2)} (\rho)}{\rho} \right|_{\rho=2\pi k_1 A}.$$

Now, recalling the analytic character of the function J_1 (see beginning of Appendix) and, more precisely, its series representation given by (A.2), it is clear that every derivative of J_1 of even order is also analytic and vanishes at zero. This implies analyticity of the function $J_1^{(2j+2p+2)} (\rho) /\rho$ and, consequently, a bound on its every derivative at the origin. Therefore, from (3.72), we deduce that

$$\left| \partial_{k_1}^{2n+1} \mathcal{S}_{p,j} \right|_{k_1=0^+} \right| \le C A^{2(n-p)-1},$$

for some constant $C > 0$, and hence (3.71) follows due to the fact that $p \ge 2$.

Step 3: Asymptotic estimates for the net moment components. Recalling that $d_1 = \pi m_1$ (according to (3.35)) and using (3.66), we obtain from (3.60)

$$(3.73) \qquad m_1 = 2 \iint_{D_A} x_1 B_3 (\mathbf{x}, h) \, \mathrm{d}^2 x + 2\pi \left(\frac{c_{1,0}}{A} + \frac{3c_{3,0} + c_{1,2}}{12A^3} \right) + \mathcal{O} \left(\frac{1}{A^5} \right).$$

Similarly, using (3.66), we rewrite (3.61)–(3.65), respectively, as

$$(3.74) \qquad -\frac{1}{6A^2} \iint_{D_A} x_1^3 B_3 (\mathbf{x}, h) \, \mathrm{d}^2 x + \frac{\pi}{8} \left(\frac{c_{1,0}}{A} - \frac{5c_{3,0} + c_{1,2}}{6A^3} \right) = \frac{d_3}{(2\pi)^3 A^2} + \mathcal{O} \left(\frac{1}{A^5} \right)$$

$$= \mathcal{O} \left(\frac{1}{A^2} \right),$$

$$(3.75) \qquad \frac{1}{120A^4} \iint_{D_A} x_1^5 B_3 (\mathbf{x}, h) \, \mathrm{d}^2 x - \frac{\pi}{192} \left(\frac{c_{1,0}}{3A} + \frac{7c_{3,0} + c_{1,2}}{8A^3} \right) = \frac{d_5}{(2\pi)^5 A^4} + \mathcal{O} \left(\frac{1}{A^5} \right)$$

$$= \mathcal{O} \left(\frac{1}{A^4} \right),$$

$$(3.76) \qquad -\frac{1}{5040A^6} \iint_{D_A} x_1^7 B_3 (\mathbf{x}, h) \, \mathrm{d}^2 x + \frac{\pi}{46080} \left(\frac{c_{1,0}}{A} + \frac{9c_{3,0} + c_{1,2}}{6A^3} \right) = \frac{d_7}{(2\pi)^7 A^6} + \mathcal{O} \left(\frac{1}{A^5} \right)$$

$$= \mathcal{O} \left(\frac{1}{A^5} \right),$$

$$(3.77) \qquad \frac{1}{362880A^8} \iint_{D_A} x_1^9 B_3 (\mathbf{x}, h) \, \mathrm{d}^2 x - \frac{\pi}{737280} \left(\frac{c_{1,0}}{7A} + \frac{11c_{3,0} + c_{1,2}}{60A^3} \right) = \frac{d_9}{(2\pi)^9 A^8} + \mathcal{O} \left(\frac{1}{A^5} \right)$$

$$= \mathcal{O} \left(\frac{1}{A^5} \right),$$

$$(3.78) \quad -\frac{1}{39916800A^{10}} \iint_{D_A} x_1^{11} B_3\left(\mathbf{x}, h\right) \mathrm{d}^2 x + \frac{\pi}{8670412800} \left(\frac{98c_{1,0}}{9A} + \frac{13c_{3,0} + c_{1,2}}{A^3}\right) = \frac{d_{11}}{(2\pi)^{11} A^{10}} + \mathcal{O}\left(\frac{1}{A^5}\right)$$

$$= \mathcal{O}\left(\frac{1}{A^5}\right).$$

While the first-order estimate for m_1 given in (2.2) follows immediately from rigorously justified (3.73), the higher-order estimates require more work. Namely, we wish to combine (3.74)–(3.78) in order to eliminate in (3.73) the terms with

$$(3.79) \qquad \qquad \widetilde{c}_{1,0} := \frac{c_{1,0}}{A}, \qquad \widetilde{c}_{3,0} := \frac{c_{3,0}}{A^3}, \qquad \widetilde{c}_{1,2} := \frac{c_{1,2}}{A^3},$$

and, at the same time, would not commit a larger error (in order of A) than that of the eliminated term.

Expressing $c_{1,0}/A$ in terms of $\mathcal{O}\left(1/A^2\right)$ quantities from (3.74) and inserting it into (3.73), we deduce the second-order estimate for m_1 given by (2.3). We note, however, that, for third or higher order estimates, identity (3.74) is not useful due to the fact that its right-hand side has an unknown quantity d_3 appearing of order $\mathcal{O}\left(1/A^2\right)$ which will block any further effort to increase the accuracy of estimates.

Derivation of the third-order estimate given by (2.5) is analogous to the previous one with the only difference that $c_{1,0}/A$ is expressed (now in terms of $\mathcal{O}\left(1/A^3\right)$ quantities) from (3.75) rather than from (3.74).

To proceed with derivation of estimates (2.7) and (2.9), it is convenient first to rewrite (3.75)–(3.78), respectively, as

$$(3.80) \qquad 7\widetilde{c}_{3,0} + \widetilde{c}_{1,2} = \frac{64}{5\pi A^4} \iint_{D_A} x_1^5 B_3\left(\mathbf{x}, h\right) \mathrm{d}^2 x - \frac{8}{3}\widetilde{c}_{1,0} + \mathcal{O}\left(\frac{1}{A^4}\right) =: \mathcal{T}_5,$$

$$(3.81) \qquad 9\widetilde{c}_{3,0} + \widetilde{c}_{1,2} = \frac{384}{7\pi A^6} \iint_{D_A} x_1^7 B_3\left(\mathbf{x}, h\right) \mathrm{d}^2 x - 6\widetilde{c}_{1,0} + \mathcal{O}\left(\frac{1}{A^5}\right) =: \mathcal{T}_7,$$

$$(3.82) \qquad 11\widetilde{c}_{3,0} + \widetilde{c}_{1,2} = \frac{2560}{21\pi A^8} \iint_{D_A} x_1^9 B_3\left(\mathbf{x}, h\right) \mathrm{d}^2 x - \frac{60}{7}\widetilde{c}_{1,0} + \mathcal{O}\left(\frac{1}{A^5}\right) =: \mathcal{T}_9,$$

$$(3.83) \qquad 13\widetilde{c}_{3,0} + \widetilde{c}_{1,2} = \frac{64512}{297\pi A^{10}} \iint_{D_A} x_1^{11} B_3\left(\mathbf{x}, h\right) \mathrm{d}^2 x - \frac{98}{9}\widetilde{c}_{1,0} + \mathcal{O}\left(\frac{1}{A^5}\right) =: \mathcal{T}_{11}.$$

It is easy to see that

$$(3.84) \qquad \frac{1}{2}\left(\mathcal{T}_5 + \mathcal{T}_9\right) = \mathcal{T}_7, \qquad \frac{1}{2}\left(\mathcal{T}_7 + \mathcal{T}_{11}\right) = \mathcal{T}_9.$$

To obtain the fourth-order estimate for m_1 given in (2.7), we shall use (3.80)–(3.82). We start by using the first equation of (3.84) (together with definitions (3.80)–(3.82)) to express $\widetilde{c}_{1,0} = c_{1,0}/A$ up to order $\mathcal{O}\left(1/A^4\right)$, namely,

$$(3.85) \qquad \frac{c_{1,0}}{A} = \widetilde{c}_{1,0} = -\frac{4}{\pi} \iint_{D_A} \left[\frac{21}{5}\left(\frac{x_1}{A}\right)^4 - 36\left(\frac{x_1}{A}\right)^6 + 40\left(\frac{x_1}{A}\right)^8\right] x_1 B_3\left(\mathbf{x}, h\right) \mathrm{d}^2 x + \mathcal{O}\left(\frac{1}{A^4}\right).$$

Second, we observe that the quantity $3c_{3,0} + c_{1,2}$ appearing in (3.73) is related to (3.80)–(3.82) as follows:

$$(3.86) \qquad \frac{3c_{3,0} + c_{1,2}}{A^3} = 3\widetilde{c}_{3,0} + \widetilde{c}_{1,2} = 4\left(\mathcal{T}_5 - \mathcal{T}_7\right) + \mathcal{T}_9$$

$$= \frac{256}{\pi} \iint_{D_A} \left[\frac{1}{5}\left(\frac{x_1}{A}\right)^4 - \frac{6}{7}\left(\frac{x_1}{A}\right)^6 + \frac{10}{21}\left(\frac{x_1}{A}\right)^8\right] x_1 B_3\left(\mathbf{x}, h\right) \mathrm{d}^2 x$$

$$+ \frac{100}{21}\widetilde{c}_{1,0} + \mathcal{O}\left(\frac{1}{A^4}\right).$$

Finally, substitution of (3.86) in (3.73) followed by the use of (3.85) furnishes (2.7).

To arrive at the fifth-order estimate given by (2.9), we shall use identities (3.81)–(3.83). The second equation of (3.84) gives

$$(3.87) \qquad \frac{c_{1,0}}{A} = \widetilde{c}_{1,0} = \frac{24}{\pi} \iint_{D_A} \left[-9\left(\frac{x_1}{A}\right)^6 + 40\left(\frac{x_1}{A}\right)^8 - \frac{392}{11}\left(\frac{x_1}{A}\right)^{10}\right] x_1 B_3\left(\mathbf{x}, h\right) \mathrm{d}^2 x + \mathcal{O}\left(\frac{1}{A^5}\right).$$

We also have an analog of (3.79), namely,

(3.88)
$$\frac{3c_{3,0} + c_{1,2}}{A^3} = 3\widetilde{c}_{3,0} + \widetilde{c}_{1,2} = 5\left(\mathcal{T}_7 - \mathcal{T}_9\right) + \mathcal{T}_{11}$$

$$= \frac{128}{\pi} \iint_{D_A} \left[\frac{15}{7}\left(\frac{x_1}{A}\right)^6 - \frac{100}{21}\left(\frac{x_1}{A}\right)^8 + \frac{56}{33}\left(\frac{x_1}{A}\right)^{10}\right] x_1 B_3\left(\mathbf{x}, h\right) \mathrm{d}^2 x$$

$$+ \frac{124}{63}\widetilde{c}_{1,0} + \mathcal{O}\left(\frac{1}{A^5}\right).$$

Therefore, inserting of (3.88) in (3.73) followed by the use of (3.87) gives (2.9).

3.3.2. *Normal component of the net moment.* Similarly to the case of tangential net moment components, we expand the integrand in (3.7) in power series in a positive neighborhood of $k_1 = 0$, but, in contrast to that previous situation, the attention will now be on the real part. This yields

(3.89)
$$\left[\mathrm{Re}\,\hat{B}_3\left(k_1, 0, h\right)\right]\Big|_{k_1=0^+} = 0,$$

(3.90)
$$\frac{1}{2}\,\partial_{k_1}^2\left[\mathrm{Re}\,\hat{B}_3\left(k_1, 0, h\right)\right]\Big|_{k_1=0^+} = -2\pi^2\left[\langle x_1 M_1\rangle + \langle\left(h - x_3\right)M_3\rangle\right]$$
$$=: d_2,$$

(3.91)
$$\frac{1}{24}\,\partial_{k_1}^4\left[\mathrm{Re}\,\hat{B}_3\left(k_1, 0, h\right)\right]\Big|_{k_1=0^+} = -\frac{4\pi^4}{3}\left[3\left\langle\left(h - x_3\right)^2 x_1 M_1\right\rangle - \left\langle x_1^3 M_1\right\rangle + \left\langle\left(h - x_3\right)^3 M_3\right\rangle - 3\left\langle\left(h - x_3\right)x_1^2 M_3\right\rangle\right]$$
$$=: d_4,$$

(3.92)
$$\frac{1}{6!}\,\partial_{k_1}^6\left[\mathrm{Re}\,\hat{B}_3\left(k_1, 0, h\right)\right]\Big|_{k_1=0^+} = -\frac{4\pi^6}{15}\left[5\left\langle\left(h - x_3\right)^4 x_1 M_1\right\rangle - 10\left\langle\left(h - x_3\right)^2 x_1^3 M_1\right\rangle + \left\langle x_1^5 M_1\right\rangle\right.$$
$$\left. + \left\langle\left(h - x_3\right)^5 M_3\right\rangle - 10\left\langle\left(h - x_3\right)^3 x_1^2 M_3\right\rangle + 5\left\langle\left(h - x_3\right)x_1^4 M_3\right\rangle\right]$$
$$=: d_6,$$

(3.93)
$$\frac{1}{8!}\,\partial_{k_1}^8\left[\mathrm{Re}\,\hat{B}_3\left(k_1, 0, h\right)\right]\Big|_{k_1=0^+} = -\frac{8\pi^8}{315}\left[7\left\langle\left(h - x_3\right)^6 x_1 M_1\right\rangle - 35\left\langle\left(h - x_3\right)^4 x_1^3 M_1\right\rangle + 21\left\langle\left(h - x_3\right)^2 x_1^5 M_1\right\rangle\right.$$
$$\left. - \left\langle x_1^7 M_1\right\rangle + \left\langle\left(h - x_3\right)^7 M_3\right\rangle - 21\left\langle\left(h - x_3\right)^5 x_1^2 M_3\right\rangle + 35\left\langle\left(h - x_3\right)^3 x_1^4 M_3\right\rangle\right.$$
$$\left. - 7\left\langle\left(h - x_3\right)x_1^6 M_3\right\rangle\right]$$
$$=: d_8,$$

(3.94)
$$\frac{1}{10!}\,\partial_{k_1}^{10}\left[\mathrm{Re}\,\hat{B}_3\left(k_1, 0, h\right)\right]\Big|_{k_1=0^+} = -\frac{4\pi^{10}}{2385}\left[9\left\langle\left(h - x_3\right)^8 x_1 M_1\right\rangle - 84\left\langle\left(h - x_3\right)^6 x_1^3 M_1\right\rangle + 126\left\langle\left(h - x_3\right)^4 x_1^5 M_1\right\rangle\right.$$
$$\left. - 36\left\langle\left(h - x_3\right)^2 x_1^7 M_1\right\rangle + \left\langle x_1^9 M_1\right\rangle + \left\langle\left(h - x_3\right)^9 M_3\right\rangle - 36\left\langle\left(h - x_3\right)^7 x_1^2 M_3\right\rangle\right.$$
$$\left. + 126\left\langle\left(h - x_3\right)^5 x_1^4 M_3\right\rangle - 84\left\langle\left(h - x_3\right)^3 x_1^6 M_3\right\rangle + 9\left\langle\left(h - x_3\right)x_1^8 M_3\right\rangle\right]$$
$$=: d_{10},$$

On the other hand, (3.10) implies that, for $n \in \mathbb{N}_0$,

(3.95)
$$\partial_{k_1}^{2n}\left[\mathrm{Re}\,\hat{B}_3\left(k_1, 0, h\right)\right]\Big|_{k_1=0^+} = (-1)^n\,(2\pi)^{2n}\iint_{D_A} x_1^{2n} B_3\left(\mathbf{x}, h\right)\mathrm{d}^2 x$$

$$+ \partial_{k_1}^{2n}\left(\iint_{\mathbb{R}^2\backslash D_A}\cos\left(2\pi k_1 x_1\right)B_3^{\mathrm{asympt}}\left(\mathbf{x}, h\right)\mathrm{d}^2 x\right)\Bigg|_{k_1=0^+}$$

$$+ \partial_{k_1}^{2n}\left(\iint_{\mathbb{R}^2\backslash D_A}\cos\left(2\pi k_1 x_1\right)\left[B_3\left(\mathbf{x}, h\right) - B_3^{\mathrm{asympt}}\left(\mathbf{x}, h\right)\right]\mathrm{d}^2 x\right)\Bigg|_{k_1=0^+}.$$

As before, we continue in 3 steps. First, we evaluate explicitly

$$(3.96) \qquad \mathcal{I}^{\cos} \equiv \mathcal{I}^{\cos}(k_1, A) := \iint_{\mathbb{R}^2 \setminus D_A} \cos(2\pi k_1 x_1) \, B_3^{\mathrm{asympt}}(\mathbf{x}, h) \, d^2 x,$$

that allow us to obtain from (3.95), a set of useful identities involving remainder terms. Second, we estimate the remainder terms, namely,

$$(3.97) \qquad \mathcal{R}_{2n}^{\cos} \equiv \mathcal{R}_{2n}^{\cos}(A) := \partial_{k_1}^{2n} \left(\iint_{\mathbb{R}^2 \setminus D_A} \cos(2\pi k_1 x_1) \left[B_3(\mathbf{x}, h) - B_3^{\mathrm{asympt}}(\mathbf{x}, h) \right] d^2 x \right) \Bigg|_{k_1 = 0^+}, \quad n \in \mathbb{N}_0,$$

for large A. At last, we rigorously derive asymptotic formulas (2.4), (2.6), (2.8) from the obtained set of identities.

Step 1: Derivation of the set of identities. Using (3.25) and passing to the polar coordinates using $x = r\cos\theta$, $y = r\sin\theta$, $d^2 x = r dr d\theta$, we obtain from (3.96)

$$(3.98) \qquad \mathcal{I}^{\cos} = c_{0,0} \mathcal{I}_1^{\cos} + c_{2,0} \mathcal{I}_2^{\cos} + c_{0,2} \mathcal{I}_3^{\cos},$$

where

$$(3.99) \qquad \mathcal{I}_1^{\cos} := \int_A^\infty \int_0^{2\pi} \cos(2\pi k_1 r \cos\theta) \, d\theta \frac{dr}{r^2},$$

$$(3.100) \qquad \mathcal{I}_2^{\cos} := \int_A^\infty \int_0^{2\pi} \cos(2\pi k_1 r \cos\theta) \cos^2\theta d\theta \frac{dr}{r^4},$$

$$(3.101) \qquad \mathcal{I}_3^{\cos} := \int_A^\infty \int_0^{2\pi} \cos(2\pi k_1 r \cos\theta) \sin^2\theta d\theta \frac{dr}{r^4}.$$

Here, we used results (A.40)–(A.41) of Lemma A.4 multiple times to deduce vanishing of the integrals associated with the terms which involve $c_{1,0}, c_{0,1}, c_{1,1}, c_{3,0}, c_{0,3}, c_{2,1}, c_{1,2}$.

Using the integral representation of Bessel functions (due to (A.4)), integration by parts, the asymptotics of J_1 given by (A.6), and the relation $J_0'(x) = -J_1(x)$ (see (A.10)), we can write, for $k_1 > 0$,

$$(3.102) \qquad \mathcal{I}_1^{\cos} = 2\pi \int_A^\infty J_0(2\pi k_1 r) \frac{dr}{r^2} = (2\pi)^2 k_1 \int_{2\pi k_1 A}^\infty \frac{J_0(x)}{x^2} dx$$
$$= (2\pi)^2 k_1 \left(\frac{J_0(\rho)}{\rho} - \int_\rho^\infty \frac{J_1(x)}{x} dx \right) \Bigg|_{\rho = 2\pi k_1 A},$$

$$(3.103) \qquad \mathcal{I}_2^{\cos} = -2\pi \int_A^\infty J_0''(2\pi k_1 r) \frac{dr}{r^4} = (2\pi)^4 k_1^3 \int_{2\pi k_1 A}^\infty \frac{J_1'(x)}{x^4} dx$$
$$= (2\pi)^4 k_1^3 \left(\frac{J_1(\rho)}{\rho^4} + 4 \int_\rho^\infty \frac{J_1(x)}{x^5} dx \right) \Bigg|_{\rho = 2\pi k_1 A},$$

$$(3.104) \qquad \mathcal{I}_3^{\cos} = 2\pi \int_A^\infty \left[J_0(2\pi k_1 r) + J_0''(2\pi k_1 r) \right] \frac{dr}{r^4} = (2\pi)^4 k_1^3 \int_{2\pi k_1 A}^\infty \frac{J_0(x) + J_0''(x)}{x^4} dx,$$
$$= (2\pi)^4 k_1^3 \left(\frac{J_0(\rho)}{3\rho^3} - \frac{J_1(\rho)}{\rho^4} - \frac{1}{3} \int_\rho^\infty \frac{J_1(x)}{x^3} dx - 4 \int_\rho^\infty \frac{J_1(x)}{x^5} dx \right) \Bigg|_{\rho = 2\pi k_1 A}.$$

Inserting here the results of Lemmas A.2–A.3, namely, (A.22), (A.34)–(A.35), we arrive at

$$(3.105) \qquad \mathcal{I}_1^{\cos} = \frac{2\pi}{A} \Bigg[2J_0(\rho) + \frac{J_1(\rho)}{\rho} - J_1'(\rho) - 2J_1(\rho) - \rho - \rho J_1(\rho)$$
$$+ \rho^2 J_0(\rho) - \frac{\pi \rho^2}{2} J_0(\rho) H_1(\rho) + \frac{\pi \rho^2}{2} J_1(\rho) H_0(\rho) \Bigg] \Bigg|_{\rho = 2\pi k_1 A},$$

$$(3.106) \qquad \mathcal{I}_2^{\cos} = \frac{2\pi}{15A^3} \Bigg[-\frac{J_0(\rho)}{3} + 4J_1'(\rho) - \frac{4\rho^2 J_0(\rho)}{3} - \frac{4\rho J_1(\rho)}{3} + \frac{4\rho^3}{3} + \frac{4\rho^3 J_1(\rho)}{3}$$
$$- \frac{4\rho^4 J_0(\rho)}{3} + \frac{2\pi \rho^4}{3} J_0(\rho) H_1(\rho) - \frac{2\pi \rho^4}{3} J_1(\rho) H_0(\rho) \Bigg] \Bigg|_{\rho = 2\pi k_1 A},$$

$$(3.107) \qquad \mathcal{I}_3^{\cos} = \frac{2\pi}{3A^3} \left[J_0(\rho) + \frac{J_1(\rho)}{15\rho} - \frac{4J_1'(\rho)}{5} - \frac{\rho^2 J_0(\rho)}{15} - \frac{\rho J_1(\rho)}{15} + \frac{\rho^3}{15} + \frac{\rho^3 J_1(\rho)}{15} \right.$$
$$\left. - \frac{\rho^4 J_0(\rho)}{15} + \frac{\pi\rho^4}{30} J_0(\rho) H_1(\rho) - \frac{\pi\rho^4}{30} J_1(\rho) H_0(\rho) \right] \Bigg|_{\rho=2\pi k_1 A}.$$

Substitution of (3.105)–(3.107) into (3.98), we employ (A.2) and (A.11) to compute

$$(3.108) \qquad \mathcal{I}^{\cos}\big|_{k_1=0^+} = 2\pi \left(\frac{c_{0,0}}{A} + \frac{11c_{2,0} + 19c_{0,2}}{90A^3} \right),$$

$$(3.109) \qquad \partial_{k_1}^2 \mathcal{I}^{\cos}\big|_{k_1=0^+} = \frac{1}{4} (2\pi)^3 A^2 \left(\frac{c_{0,0}}{A} - \frac{131c_{2,0} + 49c_{0,2}}{180A^3} \right),$$

$$(3.110) \qquad \partial_{k_1}^4 \mathcal{I}^{\cos}\big|_{k_1=0^+} = -\frac{1}{8} (2\pi)^5 A^4 \left(\frac{c_{0,0}}{A} + \frac{229c_{2,0} + 41c_{0,2}}{90A^3} \right),$$

$$(3.111) \qquad \partial_{k_1}^6 \mathcal{I}^{\cos}\big|_{k_1=0^+} = \frac{1}{16} (2\pi)^7 A^6 \left(\frac{c_{0,0}}{A} + \frac{109c_{2,0} + 11c_{0,2}}{72A^3} \right),$$

$$(3.112) \qquad \partial_{k_1}^8 \mathcal{I}^{\cos}\big|_{k_1=0^+} = -3136 (2\pi)^9 A^8 \left(\frac{c_{0,0}}{7A} + \frac{17c_{2,0} + c_{0,2}}{90A^3} \right),$$

$$(3.113) \qquad \partial_{k_1}^{10} \mathcal{I}^{\cos}\big|_{k_1=0^+} = \frac{7}{256} (2\pi)^{11} A^{10} \left(\frac{c_{0,0}}{A} + \frac{523c_{2,0} + 17c_{0,2}}{420A^3} \right).$$

Taking into account (3.96), (3.97), we use (3.89)–(3.94) and (3.108)–(3.113) in (3.95) with $n = 0, \ldots, 5$, respectively, and thus arrive at the following set of identities:

$$(3.114) \qquad \iint_{D_A} B_3(\mathbf{x}, h) \, \mathrm{d}^2 x + 2\pi \left(\frac{c_{0,0}}{A} + \frac{11c_{2,0} + 19c_{0,2}}{90A^3} \right) + \mathcal{R}_0^{\cos} = 0,$$

$$(3.115) \qquad -\frac{(2\pi)^2}{2} \iint_{D_A} x_1^2 B_3(\mathbf{x}, h) \, \mathrm{d}^2 x + \frac{(2\pi)^3}{4} A^2 \left(\frac{c_{0,0}}{A} - \frac{131c_{2,0} + 49c_{0,2}}{180A^3} \right) + \frac{1}{2}\mathcal{R}_2^{\cos} = d_2,$$

$$(3.116) \qquad \frac{(2\pi)^4}{24} \iint_{D_A} x_1^4 B_3(\mathbf{x}, h) \, \mathrm{d}^2 x - \frac{1}{192} (2\pi)^5 A^4 \left(\frac{c_{0,0}}{A} + \frac{229c_{2,0} + 41c_{0,2}}{90A^3} \right) + \frac{1}{4!}\mathcal{R}_4^{\cos} = d_4,$$

$$(3.117) \qquad -\frac{(2\pi)^6}{720} \iint_{D_A} x_1^6 B_3(\mathbf{x}, h) \, \mathrm{d}^2 x + \frac{1}{11520} (2\pi)^7 A^6 \left(\frac{c_{0,0}}{A} + \frac{109c_{2,0} + 11c_{0,2}}{72A^3} \right) + \frac{1}{6!}\mathcal{R}_6^{\cos} = d_6,$$

$$(3.118) \qquad \frac{(2\pi)^8}{40320} \iint_{D_A} x_1^8 B_3(\mathbf{x}, h) \, \mathrm{d}^2 x - \frac{7}{90} (2\pi)^9 A^8 \left(\frac{c_{0,0}}{7A} + \frac{17c_{2,0} + c_{0,2}}{90A^3} \right) + \frac{1}{8!}\mathcal{R}_8^{\cos} = d_8,$$

$$(3.119) \qquad -\frac{(2\pi)^{10}}{3628800} \iint_{D_A} x_1^{10} B_3(\mathbf{x}, h) \, \mathrm{d}^2 x + \frac{1}{132710400} (2\pi)^{11} A^{10} \left(\frac{c_{0,0}}{A} + \frac{523c_{2,0} + 17c_{0,2}}{420A^3} \right) + \frac{1}{10!}\mathcal{R}_{10}^{\cos} = d_{10}.$$

Step 2: Analysis of the remainder terms $\mathcal{R}_{2n}^{\cos}$ for $0 \leq n \leq 5$. We shall show that

$$(3.120) \qquad \mathcal{R}_{2n}^{\cos} = \mathcal{O}\left(\frac{1}{A^{5-2n}} \right), \qquad 0 \leq n \leq 5.$$

The reasoning will we identical to that of Step 2 of Subsection 3.3.1, therefore, we omit repetition of some details.

Using previously introduced notation L_N (see (3.67)), we can write

$$(3.121) \qquad \mathcal{R}_{2n}^{\cos} = \partial_{k_1}^{2n} \left(\iint_{\mathbb{R}^2 \setminus D_A} \cos(2\pi k_1 x_1) L_N(\mathbf{x}) \, \mathrm{d}^2 x \right) \Bigg|_{k_1=0^+}$$
$$+ \partial_{k_1}^{2n} \left(\iint_{\mathbb{R}^2 \setminus D_A} \cos(2\pi k_1 x_1) \left[B_3(\mathbf{x}, h) - B_3^{\mathrm{asympt}}(\mathbf{x}, h) - L_N(\mathbf{x}) \right] \mathrm{d}^2 x \right) \Bigg|_{k_1=0^+}.$$

Assuming $N > 9$, we estimate

$$\left| \partial_{k_1}^{2n} \left(\iint_{\mathbb{R}^2 \setminus D_A} \cos\left(2\pi k_1 x_1\right) \left[B_3\left(\mathbf{x}, h\right) - B_3^{\text{asympt}}\left(\mathbf{x}, h\right) - L_N\left(\mathbf{x}\right) \right] \mathrm{d}^2 x \right) \Big|_{k_1 = 0^+} \right|$$

$$= \left| \left(\iint_{\mathbb{R}^2 \setminus D_A} \frac{\left(2\pi x_1\right)^{2n}}{|\mathbf{x}|^{N+4}} \, |\mathbf{x}|^{N+4} \left[B_3\left(\mathbf{x}, h\right) - B_3^{\text{asympt}}\left(\mathbf{x}, h\right) - L_N\left(\mathbf{x}\right) \right] \mathrm{d}^2 x \right) \right|$$

$$\leq \left(2\pi\right)^{2n+1} \widetilde{C}_N \int_A^\infty \frac{\mathrm{d}r}{r^{N-2n+2}} = \frac{\left(2\pi\right)^{2(n+1)} \widetilde{C}_N}{\left(N - 2n + 1\right)} \frac{1}{A^{N-2n+1}},$$

for some constant $\widetilde{C}_N > 0$, and note that this complies with (3.120).

We now fix $N = 10$ and proceed with estimating the term on the first line of (3.120). Identities (A.40)–(A.41) of Lemma A.4 entail that the only non-vanishing terms are those of the form

$$(3.122) \qquad \int_A^\infty \int_0^{2\pi} \cos\left(2\pi k_1 r \cos\theta\right) \cos^{2(p-l)}\theta \, \cos^{2l}\theta \mathrm{d}\theta \frac{\mathrm{d}r}{r^{2(p+1)}}, \qquad 0 \leq l \leq p, \quad p \geq 2,$$

and hence it is sufficient to only deal with the quantities

$$(3.123) \qquad \mathcal{C}_{p,j} := \int_A^\infty \int_0^{2\pi} \cos\left(2\pi k_1 r \cos\theta\right) \cos^{2j}\theta \, \mathrm{d}\theta \frac{\mathrm{d}r}{r^{2(p+1)}}, \qquad 0 \leq j \leq p, \quad p \geq 2.$$

As before, p in (3.122)–(3.123) is related to q from (3.67) as $q = 2p + 1$, $p \geq 2$: only the terms $\mathcal{O}\left(1/A^7\right)$, $\mathcal{O}\left(1/A^9\right)$, ..., in (3.67) contribute to $\mathcal{R}_{2n}^{\cos}$. Hence, we consider $\mathcal{C}_{p,j}$ only for $p \leq \left\lfloor \frac{N-1}{2} \right\rfloor = 8$.

We are intending to show that the following estimate holds for all $0 \leq n \leq 5$, $0 \leq j \leq p$, $2 \leq p \leq 8$:

$$(3.124) \qquad \partial_{k_1}^{2n} \mathcal{C}_{p,j} \big|_{k_1 = 0^+} = \mathcal{O}\left(\frac{1}{A^{5-2n}}\right).$$

For $0 \leq n \leq p$, we have

$$\partial_{k_1}^{2n} \mathcal{C}_{p,j} \big|_{k_1 = 0^+} = \left(-1\right)^n \left(2\pi\right)^{2n} \int_A^\infty \int_0^{2\pi} \cos^{2j}\theta \, \mathrm{d}\theta \frac{\mathrm{d}r}{r^{2(p-n)+2}},$$

and hence

$$\left| \partial_{k_1}^{2n+1} \mathcal{C}_{p,j} \big|_{k_1 = 0^+} \right| \leq \left(2\pi\right)^{2n+1} \int_A^\infty \frac{\mathrm{d}r}{r^{2(p-n)+2}} = \mathcal{O}\left(\frac{1}{A^{2(p-n)+1}}\right),$$

where the convergence of the last integral is due to $p \geq n$. The obtained estimate satisfies (3.124) due to $p \geq 2$.

For $n \geq p + 1$, we use (A.4) to rewrite (3.123) as

$$\mathcal{C}_{p,j} = \left(-1\right)^j 2\pi \int_A^\infty J_0^{(2j)}\left(2\pi k_1 r\right) \frac{\mathrm{d}r}{r^{2(p+1)}}, \qquad 0 \leq j \leq p, \quad p \geq 2.$$

We then evaluate

$$(3.125) \qquad \partial_{k_1}^{2n} \mathcal{C}_{p,j} = \left(-1\right)^j \left(2\pi\right)^{2(p+1)} \partial_{k_1}^{2(n-p)-1} \int_A^\infty J_0^{(2j+2p+1)}\left(2\pi k_1 r\right) \frac{\mathrm{d}r}{r}$$

$$= \left(-1\right)^{j+1} \left(2\pi\right)^{2p+3} A \partial_{k_1}^{2(n-p)-2} \frac{J_0^{(2j+2p+1)}\left(\rho\right)}{\rho} \Bigg|_{\rho = 2\pi k_1 A}$$

$$= \left(-1\right)^{j+1} \left(2\pi\right)^{2n+1} A^{2(n-p)-1} \frac{\mathrm{d}^{2(n-p)-2}}{\mathrm{d}\rho^{2(n-p)-2}} \left(\frac{J_0^{(2j+2p+1)}\left(\rho\right)}{\rho} \right) \Bigg|_{\rho = 2\pi k_1 A},$$

where the following identity, for $k_1 > 0$, was used:

$$\partial_{k_1} \int_A^\infty J_0^{(2j+2p+1)}\left(2\pi k_1 r\right) \frac{\mathrm{d}r}{r} = \partial_{k_1} \int_{2\pi k_1 A}^\infty J_0^{(2j+2p+1)}\left(\rho\right) \frac{\mathrm{d}\rho}{\rho}$$

$$= -2\pi A \frac{J_0^{(2j+2p+1)}\left(\rho\right)}{\rho} \Bigg|_{\rho = 2\pi k_1 A}.$$

Due to analyticity J_0 (see beginning of Appendix), we see from (A.2) that its every derivative of odd order is also analytic and vanishes at zero. Hence, $J_0^{(2j+2p+1)}(\rho)/\rho$ is analytic as well, and, in particular, has bounded derivatives at the origin. Therefore, (3.125) entails that

$$\left| \partial_{k_1}^{2n} \mathcal{C}_{p,j} \big|_{k_1=0^+} \right| \leq \widetilde{C} A^{2(n-p)-1},$$

for some constant $\widetilde{C} > 0$, and hence (3.124) follows due to the fact that $p \geq 2$.

Step 3: Asymptotic estimates for the net moment component. Since, according to (3.26), we have $c_{0,0} = -\frac{m_3}{4\pi}$, and hence using (3.120), it follows from (3.114) that

$$(3.126) \qquad m_3 = 2 \iint_{D_A} B_3(\mathbf{x}, h)\, \mathrm{d}^2 x + \frac{\pi}{45A^2}(11c_{2,0} + 19c_{0,2}) + \mathcal{O}\left(\frac{1}{A^4}\right).$$

Employing (3.66), we rewrite (3.61)–(3.65), respectively, as

$$(3.127) \qquad -\frac{1}{2A^2} \iint_{D_A} x_1^2 B_3(\mathbf{x}, h)\, \mathrm{d}^2 x - \frac{\pi}{8}\left(\frac{m_3}{\pi A} + \frac{131c_{2,0} + 49c_{0,2}}{45A^3}\right) = \frac{d_2}{(2\pi A)^2} + \mathcal{O}\left(\frac{1}{A^5}\right)$$
$$= \mathcal{O}\left(\frac{1}{A^2}\right),$$

$$(3.128) \qquad \frac{1}{24A^4} \iint_{D_A} x_1^4 B_3(\mathbf{x}, h)\, \mathrm{d}^2 x + \frac{\pi}{192}\left(\frac{m_3}{2\pi A} - \frac{229c_{2,0} + 41c_{0,2}}{45A^3}\right) = \frac{d_4}{(2\pi A)^4} + \mathcal{O}\left(\frac{1}{A^5}\right)$$
$$= \mathcal{O}\left(\frac{1}{A^4}\right),$$

$$(3.129) \qquad -\frac{1}{720A^6} \iint_{D_A} x_1^6 B_3(\mathbf{x}, h)\, \mathrm{d}^2 x - \frac{\pi}{2}\left(\frac{m_3}{\pi A} - \frac{109c_{2,0} + 11c_{0,2}}{18A^3}\right) = \frac{d_6}{(2\pi A)^6} + \mathcal{O}\left(\frac{1}{A^5}\right)$$
$$= \mathcal{O}\left(\frac{1}{A^5}\right),$$

$$(3.130) \qquad \frac{1}{40320A^8} \iint_{D_A} x_1^8 B_3(\mathbf{x}, h)\, \mathrm{d}^2 x + \frac{\pi}{147456}\left(\frac{m_3}{14\pi A} - \frac{17c_{2,0} + c_{0,2}}{45A^3}\right) = \frac{d_8}{(2\pi A)^8} + \mathcal{O}\left(\frac{1}{A^5}\right)$$
$$= \mathcal{O}\left(\frac{1}{A^5}\right),$$

$$(3.131) \qquad -\frac{1}{3628800A^{10}} \iint_{D_A} x_1^{10} B_3(\mathbf{x}, h)\, \mathrm{d}^2 x - \frac{\pi}{66355200}\left(\frac{m_3}{\pi A} - \frac{523c_{2,0} + 17c_{0,2}}{105A^3}\right) = \frac{d_{10}}{(2\pi A)^{10}} + \mathcal{O}\left(\frac{1}{A^5}\right)$$
$$= \mathcal{O}\left(\frac{1}{A^5}\right).$$

We see that (3.126) already provides the second-order estimate of m_3 given in (2.4). The higher-order estimates given by (2.6) and (2.8) can be obtained by combining (3.128)–(3.130). Note that elimination of m_3 using either (3.127) would incur a rather large error $\mathcal{O}(1/A)$, reducing the final estimate to the first order. Similarly, any use of (3.131) would result in an estimate of order $\mathcal{O}(1/A^4)$, but such an estimate could already be deduced using the other listed above relations.

For the sake of simplification, let us set

$$(3.132) \qquad \widetilde{c}_{2,0} := \frac{c_{2,0}}{A^3}, \qquad \widetilde{c}_{0,2} := \frac{c_{0,2}}{A^3},$$

and rewrite (3.126) and (3.128)–(3.130), respectively, as

$$(3.133) \qquad 11\widetilde{c}_{2,0} + 19\widetilde{c}_{0,2} = -\frac{45}{\pi} \iint_{D_A} B_3(\mathbf{x}, h)\, \mathrm{d}^2 x + \frac{45m_3}{2\pi A} + \mathcal{O}\left(\frac{1}{A^5}\right) =: \mathcal{T}_0,$$

$$(3.134) \qquad 131\widetilde{c}_{2,0} + 49\widetilde{c}_{0,2} = -\frac{180}{\pi A^2} \iint_{D_A} x_1^2 B_3(\mathbf{x}, h)\, \mathrm{d}^2 x - \frac{45m_3}{\pi A} + \mathcal{O}\left(\frac{1}{A^2}\right) =: \mathcal{T}_2,$$

$$(3.135) \qquad 229\widetilde{c}_{2,0} + 41\widetilde{c}_{0,2} = \frac{360}{\pi A^4} \iint_{D_A} x_1^4 B_3(\mathbf{x}, h)\, \mathrm{d}^2 x + \frac{45m_3}{2\pi A} + \mathcal{O}\left(\frac{1}{A^4}\right) =: \mathcal{T}_4,$$

j	1	2	3	4
$x_1^{(j)}$	$3.5 \cdot 10^{-5}$ m	0 m	$4.0 \cdot 10^{-5}$ m	$-4.0 \cdot 10^{-5}$ m
$x_2^{(j)}$	$3.0 \cdot 10^{-5}$ m	0 m	$-5.5 \cdot 10^{-5}$ m	$5.5 \cdot 10^{-5}$ m
$x_3^{(j)}$	$1.0 \cdot 10^{-5}$ m	$7.0 \cdot 10^{-5}$ m	$11.5 \cdot 10^{-5}$ m	$2.5 \cdot 10^{-5}$ m
$m_1^{(j)}$	$4.5 \cdot 10^{-12}$ A $\cdot$ m^2	$2.5 \cdot 10^{-12}$ A $\cdot$ m^2	$-3.0 \cdot 10^{-12}$ A $\cdot$ m^2	$-1.0 \cdot 10^{-12}$ A $\cdot$ m^2
$m_2^{(j)}$	$3.5 \cdot 10^{-12}$ A $\cdot$ m^2	$4.5 \cdot 10^{-12}$ A $\cdot$ m^2	$2.0 \cdot 10^{-12}$ A $\cdot$ m^2	$2.0 \cdot 10^{-12}$ A $\cdot$ m^2
$m_3^{(j)}$	$1.0 \cdot 10^{-12}$ A $\cdot$ m^2	$0.5 \cdot 10^{-12}$ A $\cdot$ m^2	$2.5 \cdot 10^{-12}$ A $\cdot$ m^2	$1.5 \cdot 10^{-12}$ A $\cdot$ m^2

TABLE 1. Positions and dipolar moments of the synthetic magnetisation distribution (4 dipoles)

$$(3.136) \qquad 109\widetilde{c}_{2,0} + 11\widetilde{c}_{0,2} = \frac{576}{\pi A^6} \iint_{D_A} x_1^6 B_3(\mathbf{x}, h)\, \mathrm{d}^2 x + \frac{18 m_3}{\pi A} + \mathcal{O}\left(\frac{1}{A^5}\right) =: \mathcal{T}_6,$$

$$(3.137) \qquad 17\widetilde{c}_{2,0} + \widetilde{c}_{0,2} = \frac{1152}{7\pi A^8} \iint_{D_A} x_1^8 B_3(\mathbf{x}, h)\, \mathrm{d}^2 x + \frac{45 m_3}{14\pi A} + \mathcal{O}\left(\frac{1}{A^5}\right) =: \mathcal{T}_8.$$

There are multiple ways to combine equations (3.133)–(3.137) to eliminate $\widetilde{c}_{2,0}$ and $\widetilde{c}_{0,2}$. In particular, we should use the following two relations which are straightforward to verify:

$$(3.138) \qquad \mathcal{T}_0 = \mathcal{T}_4 - 2\mathcal{T}_6, \qquad \mathcal{T}_0 = 4\mathcal{T}_6 - 25\mathcal{T}_8.$$

Substitution of (3.133), (3.135) and (3.136) in the first relation of (3.138) allows us to solve for m_3/A. This leads to the third-order estimate of m_3 given by (2.6).

Similarly, plugging (3.133), (3.136) and (3.137) into the second relation of (3.138) gives (2.8), a fourth-order estimate of m_3.

4. NUMERICAL VALIDATION AND PRACTICAL CONSIDERATIONS

We demonstrate results by performing a numerical simulation on a synthetic example. In this example, we choose magnetisation distribution to consist of 4 magnetic dipoles: $\vec{\mathcal{M}}(\vec{x}) = \sum_{j=1}^{4} \vec{m}_j \delta(\vec{x} - \vec{x_j})$ with δ denoting the Dirac delta function. The positions and the components of dipolar moments of each dipole are given in Table 1.

The net moment of this magnetisation distribution is equal to

$$(4.1) \qquad \vec{m}^{\text{true}} = \sum_{j=1}^{4} \vec{m}^{(j)} = (3.0, 12.0, 5.5)^T\, 10^{-12}\ \text{A} \cdot \text{m}^2.$$

The produced magnetic field B_3 is given by

$$(4.2)\ B_3(\mathbf{x}, h) = \frac{\mu_0}{4\pi} \sum_{j=1}^{4} \frac{3\left(h - x_3^{(j)}\right)\left[\left(x_1 - x_1^{(j)}\right) m_1^{(j)} + \left(x_2 - x_2^{(j)}\right) m_2^{(j)}\right] + \left(2\left(h - x_3^{(j)}\right)^2 - \left|\mathbf{x} - \mathbf{x}^{(j)}\right|^2\right) m_3^{(j)}}{\left(\left|\mathbf{x} - \mathbf{x}^{(j)}\right|^2 + \left(h - x_3^{(j)}\right)^2\right)^{5/2}},$$

and is measured on the disk $D_A = \left\{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < A\right\}$ at the height $x_3 = h = 2.5 \cdot 10^{-4}$ m. Since we now work in Si units, we should recall Remark 2.2 and take into account the previously omitted factor $\mu_0 = 4\pi \cdot 10^{-7}$ N / A^2.

In order to check robustness of the moment estimates obtained in Theorem 2.1, we also perform simulations on data with a synthetic noise. Namely, we modify B_3 using additive Gaussian white noise with the amplitude $\sqrt{10^{-\text{SNR}/10} \cdot \text{Var}(B_3)}$, where SNR is the signal-to-noise ratio (in decibels) and Var (B_3) is the variance of B_3 on D_A. For our simulations, we choose SNR = 20 dB which corresponds to the 10% noise level.

In Figure 4.1, we illustrate the field (4.2) and its noise component on the disk D_A of radius $A = 7.5 \cdot 10^{-4}$ m.

We shall now compute the integrals on the right-hand sides of (2.2)–(2.9) for different values of A. According to our asymptotic result for large A, we expect to see that, as A grows, the values of each of these integrals converge, with a different rate, to the value of a component of the net moment given by (4.1). Figures 4.2–4.3 show exactly that for the tangential and normal net moment components, respectively. We note that in Figure 4.3 and further figures involving the normal net moment component m_3, a pair of the estimates are used: one with $x_j = x_1$ in (2.6), (2.8) and the other with $x_j = x_2$.

To illustrate better the variability of the convergence rates for different asymptotic estimates, in Figures 4.4–4.5, we plot the differences $\left|m_j^{\text{true}} - m_j\right|$, $j \in \{1, 2, 3\}$, against A in logarithmic scale.
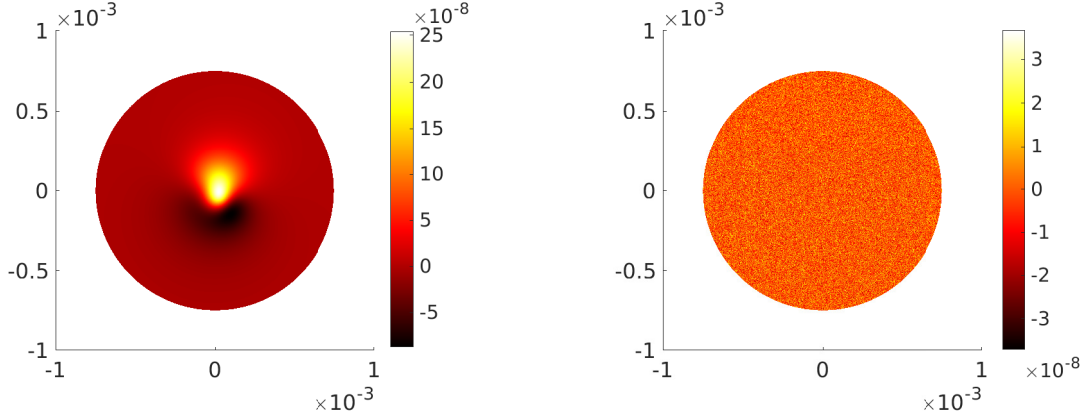
FIGURE 4.1. Magnetic field $B_3(\mathbf{x}, h)$ (left) and added noise (right) on D_A for $A = 7.5 \cdot 10^{-4}$ m.
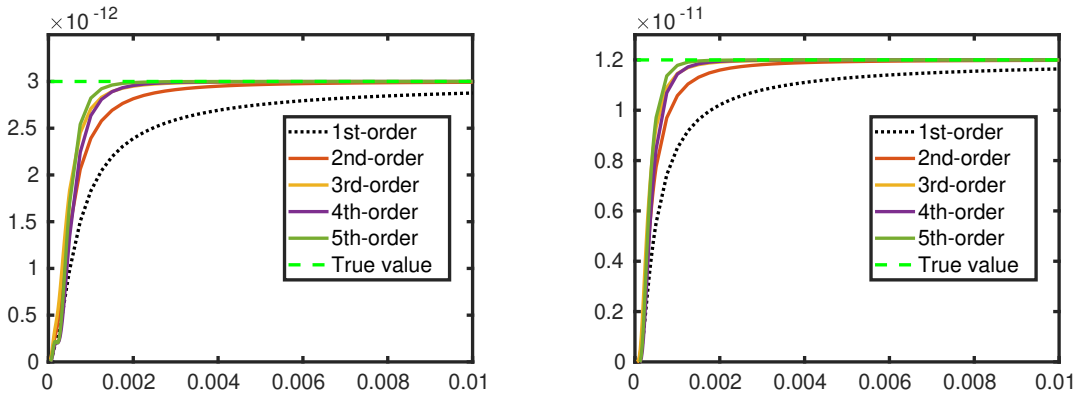
FIGURE 4.2. Estimates of the tangential net moment components m_1 (left) and m_2 (right) versus A.

Finally, in Figure 4.6, we directly test the estimates of the net moment components when the magnetic field is contaminated by noise (with the noise model described above). We observe the persistence of the lower-order estimates for the net moment components whereas the higher-order estimates clearly perform significantly worse.

Let us now briefly comment on the fact that we chose to illustrate the results on a magnetisation distribution with a singular support. Besides its simplicity, this choice is physically motivated as any magnetisation can be thought of a combination of dipole sources. From mathematical (numerical) viewpoint, a magnetic field produced by continuous magnetisation distribution is given by the integral whose numerical approximation (quadrature rule) is nothing but a weighted sum of dipoles. Consequently, we do not expect results for continuous magnetisation distributions to be of any drastical difference. On the other hand, this highlights the applicability of our methodology to the magnetisations that could be much more singular than smooth or square-integrable functions.

5. DISCUSSION AND CONCLUSION

Motivated by a concrete experimental set-up, we considered a problem of estimating net magnetisation of a sample from one component of the magnetic field available in the limited measurement area in the plane above the sample. We approached this problem asymptotically, assuming the size of the measurement area to be large. We derived a set of explicit formulas for the asymptotic estimates of all three components of the net moment. For simplicity, we considered only the case of the circular geometry, i.e., where the measurement area is a disk. Analogous results to those in Theorem 2.1 could be deduced by our method also for the rectangular geometry which technically is even closer to what is used in Paleomagnetism lab at EAPS department of MIT, USA. The main difference in obtaining such results with the present approach would be a technique of the asymptotic estimation of Fourier integrals (such as the second term in the right-hand side of (3.10)) which involve both small and large parameters (and hence must likely rely on a partial availability of explicit integration formulas).

In this paper, we have obtained and proved asymptotic estimates up to order 5 for the tangential net moment components m_1, m_2 and up to order 4 for the normal component m_3. The main purpose was, however, to introduce
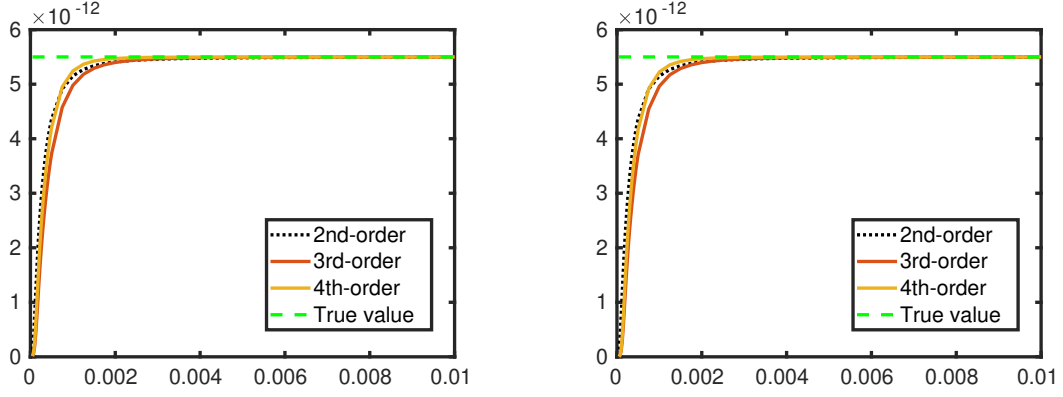
FIGURE 4.3. Estimates of the normal net moment component m_3 versus A: using x_1 (left) and x_2 (right) formulas.
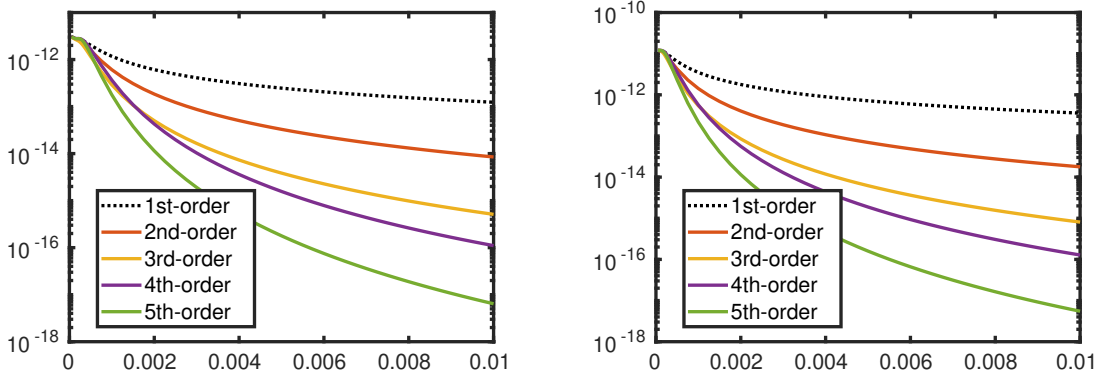
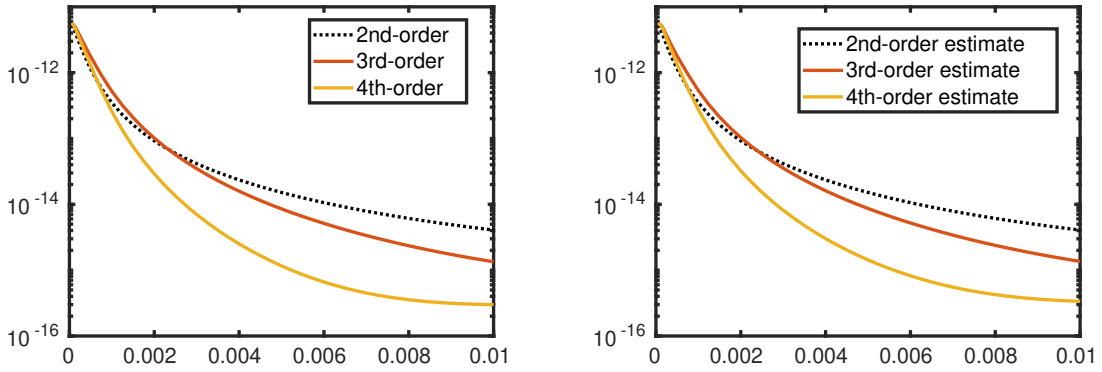FIGURE 4.4. Convergence for the estimates of m_1 (left) and m_2 (right).

FIGURE 4.5. Convergence for the m_3 estimates: using x_1 (left) and x_2 (right) formulas.

a machinery that can generate asymptotic estimates of an arbitrary order. It is clear from the proof of Theorem 2.1 and auxiliary computations in Appendix that the asymptotic order of the estimates can be upgraded by proceeding in the established manner. This would require integration of the field against polynomials of a higher order. Practical advantages of it, however, are not yet obvious. First, such estimates are going to be extremely sensitive to the presence of noise in B_3: it was demonstrated in Section 4 that while the lower- and mid-order estimates lead to the expected results, the estimates of higher orders are much more prone to the instability. This is, of course, not surprising since, as it was mentioned, the moment recovery is a severely ill-posed problem (see Section 1 and [2]), and an order of the asymptotic estimate plays the role of a regularisation parameter here. Second, when pursuing
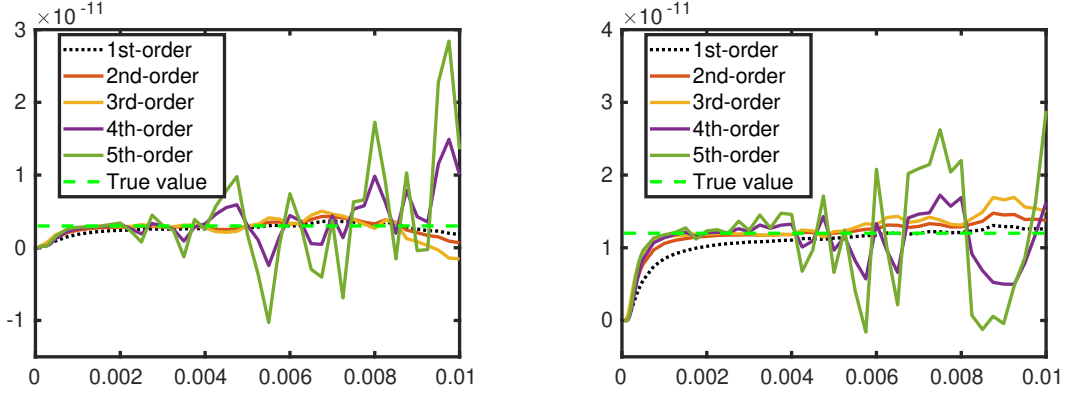
FIGURE 4.6. Estimates of the tangential net moment components m_1 (left) and m_2 (right) versus A. Noisy data.
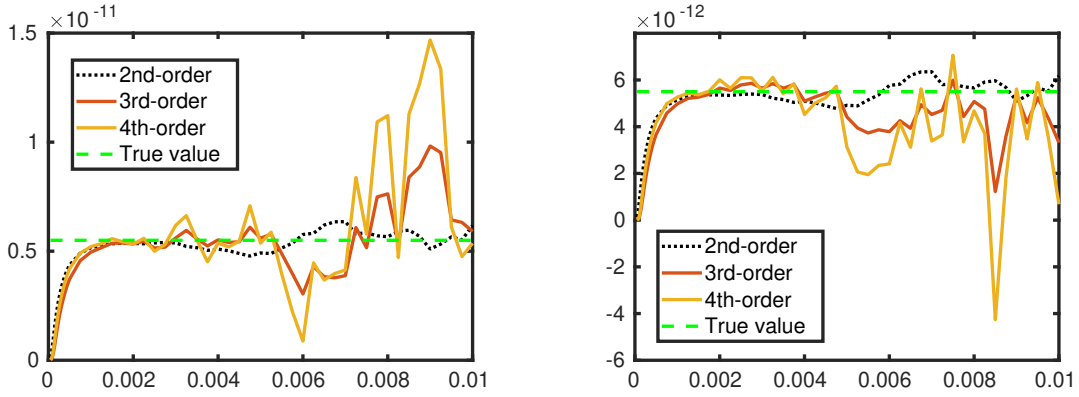

FIGURE 4.7. Estimates of the normal net moment component m_3 versus A: using x_1 (left) and x_2 (right) formulas. Noisy data.

an estimate of a higher order, one should not forget that the obtained result is of only asymptotic nature: while a high-order estimate would be advantageous for very large values of A, it may not be so for a smaller A due to a potentially large value of a multiplicative constant (in A) in the remainder term. In particular, while Figures 4.2 and 4.4 show uniform improvement over the whole range of A when using estimates of higher order for m_1 and m_3, Figures 4.3 and 4.5 demonstrate that for some small pre-asymptotic range of A, an estimate for m_3 of a lower (second) order can be better than that of a higher (third) order. A relevant issue to bear in mind is that, apart from the basic asymptoticness condition given by (2.1) (see also (2.11)), proceeding to higher-order estimates assumes implicitly (but it is evident from the form of the remainder terms) that the magnetisation is sufficiently localised so that its higher algebraic moments $L^{(n)}_{j_1,j_2,j_3} := \left\langle x_1^{j_1} x_2^{j_2} x_3^{j_3} M_n \right\rangle$, $n \in \{1,2,3\}$, (see (3.1)) do not grow too fast with respect to their order $j := j_1 + j_2 + j_3$, namely, that the quantities $L^{(n)}_{j_1,j_2,j_3}/A^j$ are not large for the value of A in question. A conclusion to draw from these observations is that the mid-order estimates (e.g., those of orders 2 and 3) are perhaps the best: both from practical prospective of proximity to the true value of the net moment and from the viewpoint of the robustness to imperfect measurements. One can also consider a possibility of choosing an estimate which is best possible for a given magnitude of A. This resembles a problem of finding an optimal truncation of asymptotic series.

In future works, we should study relation of our asymptotic results to stable estimates of the net moment using constrained optimisation approaches, in particular those obtained in [2] for planar L^2 magnetisations without the asymptotic assumption. The shape of auxiliary functions to be integrated against the measured field to produce the net moment estimates looks similar to ours. For example, for estimates of m_1 of second order and higher in Theorem 2.1, this auxiliary function is seen to consist of the main trend (the $2x_1$ term) and a correction term (a polynomial depending in the "stretched" variable x_1/A) which is the largest towards the border of the disk. It would be curious to derive an explicit asymptotic from an integro-differential equation of [2] governing the optimal solution

in a particular class. Studying other ways of stabilisation of the estimates and regularisation is also important. In this respect, it is natural to ask a question: what is the best way of using the redundancy of the set of asymptotic formulas for the net moment components to arrive at an estimate with the minimal effect of noise or to have the fastest (non-asymptotic) convergence to the true value of a net moment? For instance, in the present approach, we have a natural redundancy for all estimates of m_3 of order 3 and higher due to the freedom of choice $x_j = x_1$ and $x_j = x_2$ in formulas (2.6), (2.8) and so on. In Figure 4.3, these estimates give almost indistinguishable results (it can be checked that the difference between the two is not zero but very small), and hence may not look directly useful. However, since having more relations than unknown quantities is always better when dealing with noise, this is still an advantage. Also, it is clear from Step 3 of the proof of Theorem 2.1 that, upon involving higher order polynomials, a richer set of lower-order estimates could be obtained.

The form of asymptotic expressions suggests the direct use of polynomial (in a stretched variables x_j/A, $j \in \{1, 2\}$) ansatz for the derivation of estimates of higher order, with a potential use of deep learning (an idea suggested by an anonymous reviewer).

Another path for a possible future work is to explore the possibility of obtaining asymptotic expansions analogous to those given in Theorem 2.1 but relying on the smallness of a slghtly different asymptotic parameter. Namely, if, instead of $\left(x_1^2 + x_2^2\right)^{-3/2}$ and $\left(x_1^2 + x_2^2\right)^{-5/2}$ in (3.21) and (3.22), respectively, one factored out $\left(x_1^2 + x_2^2 + h^2\right)^{-3/2}$ and $\left(x_1^2 + x_2^2 + h^2\right)^{-5/2}$ and proceeded with appropriate modifications, the final asymptotic results would have a larger area of validity than that described by (2.1). In particular, this would cover a reduction to a dipolar case in a situation when the measurement area D_A is not necessarily large but the value of h is. However, since in the mentioned experimental set-up, the height h is small, this modification would yield a little practical benefit but would complicate analysis of Fourier integrals.

Along the same lines, it should be better understood why the height parameter h does not enter any of the final estimates in the asymptotic regime. If this remains true for estimates of any order, can this be used, for example, as a shortcut to generate the higher-order estimates?

Finally, the results of this work naturally connect to the issue of the asymptotic field extension. Indeed, the asymptotic field expansion at infinity (3.24) is seen to feature m_3 at the leading order and quantities $c_{1,0}$, $c_{0,1}$ at the next order. While the estimates of m_3 are given, to a different order, in (2.4), (2.6) and (2.8), it is evident from the proof of Theorem 2.1, that the quantities $c_{1,0}$, $c_{0,1}$ can also be asymptotically estimated, see, e.g., (3.85) and (3.87) (and their versions with x_1 replaced by x_2, as well as their lower-order analogs). Thus, this furnishes an explicit 3-term expansion of B_3 at infinity which, in general, when solving inverse magnetisation problems, should serve as a better alternative to a simple prolongation of the field by zero outside of the measurement area. Such a strategy can also potentially be used in setting up an iterative scheme. Of course, all of this is meaningful only when the actual measurement area is already large enough so that the asymptotic estimates for m_3, $c_{1,0}$, $c_{0,1}$ are sufficiently accurate.

Acknowledgements

Appendix A

We collect here several useful results about cylindrical functions and some relevant integrals. In what follows we will use the notation $\mathbb{N}_+$ to denote natural numbers, $\mathbb{N}_0 := \mathbb{N}_+ \cup \{0\}$, and the notation $\mathbb{Z}$ for integer numbers.

Basic facts about Bessel, Neumann and Struve functions. The Bessel function J_n of order $n \in \mathbb{Z}$ is an entire function satisfying the differential equation [19, (10.2.1)]

$$(A.1) \qquad z^2 J_n''(z) + z J_n'(z) + \left(z^2 - n^2\right) J_n(z) = 0, \quad z \in \mathbb{C}.$$

We have the following series expansion [19, (10.2.2)]

$$(A.2) \qquad J_n(z) = \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \, \Gamma(n+k+1)} \left(\frac{z}{2}\right)^{2k},$$

which, due to the entire character of J_n, is absolutely convergent for every $z \in \mathbb{C}$. Here, Γ denotes the Euler gamma function, for which we have, in particular, $\Gamma(k) = (k-1)!$ for $k \in \mathbb{N}_+$. Moreover, it is worth noting that $1/\Gamma(k) = 0$

for $k \in \mathbb{Z} \backslash \mathbb{N}_+$, which implies vanishing of negative powers of z in expansion (A.2) even for negative orders n. The following integral representation holds [19, (10.9.1)]

$$(A.3) \qquad J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(nt - x \sin t)\, dt, \quad x \in \mathbb{R}.$$

In particular,

$$(A.4) \qquad J_0(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin t)\, dt = \frac{1}{\pi} \int_0^\pi \cos(x \cos t)\, dt = \frac{1}{2\pi} \int_0^{2\pi} \cos(x \cos t)\, dt, \quad x \in \mathbb{R},$$

$$(A.5) \qquad J_1(x) = \frac{1}{\pi} \int_0^\pi \cos(t - x \sin t)\, dt = \frac{1}{2\pi} \int_0^{2\pi} \sin(x \cos t) \cos t\, dt, \quad x \in \mathbb{R}.$$

For $x \gg 1$, $n \in \mathbb{N}_0$, the leading order asymptotics reads [19, (10.17.2)]

$$(A.6) \qquad J_n(x) = \left(\frac{2}{\pi x} \right)^{1/2} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4} \right) + \mathcal{O}\left(\frac{1}{x^{3/2}} \right),$$

where the estimate of the remainder term is due to the discussion in [19, Sect. 10.17(iii)]. The Bessel functions J_n, $n \in \mathbb{Z}$, satisfy the connection formula [19, (10.4.1)]

$$(A.7) \qquad J_{-n}(x) = (-1)^n J_n(x),$$

as well as simple recurrence relations [19, (10.6.1)]

$$(A.8) \qquad \frac{1}{x} J_n(x) = \frac{1}{2n} (J_{n-1}(x) + J_{n+1}(x)), \quad n \neq 0,$$

$$(A.9) \qquad J_n'(x) = \frac{1}{2} (J_{n-1}(x) - J_{n+1}(x)).$$

In particular, (A.7) and (A.9) entail that

$$(A.10) \qquad J_0'(x) = -J_1(x).$$

The Struve function H_n of order $n \in \mathbb{N}_0 \cup \{-1\}$ is an entire function defined by the absolutely convergent power series [19, (11.2.1)]

$$(A.11) \qquad H_n(z) = \left(\frac{z}{2} \right)^{n+1} \sum_{k=0}^\infty \frac{(-1)^k}{\Gamma\left(k + \frac{3}{2} \right) \Gamma\left(k + n + \frac{3}{2} \right)} \left(\frac{z}{2} \right)^{2k}.$$

The companion Struve function K_n of order $n \in \mathbb{Z}$ is defined [19, (11.2.5)] as

$$(A.12) \qquad K_n(z) = H_n(z) - Y_n(z),$$

where Y_n is the Neumann function.
For $x \gg 1$, $n \in \mathbb{N}_0$, the following asymptotics hold true [19, (11.6.1)]

$$(A.13) \qquad K_n(x) = \frac{1}{\pi} \frac{2^{n+1} n!}{(2n)!} x^{n-1} + \mathcal{O}\left(x^{n-3} \right),$$

$$(A.14) \qquad Y_n(x) = \left(\frac{2}{\pi x} \right)^{1/2} \sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4} \right) + \mathcal{O}\left(\frac{1}{x^{3/2}} \right),$$

and the remainder terms are discussed in [19, Sect. 11.6(i)] and [19, Sect. 10.17(iii)], respectively. The asymptotic behavior of $H_n(x)$ for $x \gg 1$, $n \in \mathbb{N}_0$, hence follows from (A.12)–(A.14). In particular, for $x \gg 1$,

$$(A.15) \qquad H_0(x) = \left(\frac{2}{\pi x} \right)^{1/2} \sin\left(x - \frac{\pi}{4} \right) + \mathcal{O}\left(\frac{1}{x} \right),$$

$$(A.16) \qquad H_1(x) = \frac{2}{\pi} + \left(\frac{2}{\pi x} \right)^{1/2} \sin\left(x - \frac{3\pi}{4} \right) + \mathcal{O}\left(\frac{1}{x^{3/2}} \right).$$

Moreover, we have the following connection formula

$$(A.17) \qquad H_{-1}(x) = \frac{2}{\pi} - H_1(x).$$

Some useful integrals pertinent to the cylindrical functions. The following lemmas establish several integral relations which are also crucial for the proof in Section 3.

Lemma A.1. *For $n \in \mathbb{N}_+$, $\rho > 0$, the following identity holds*

$$(A.18) \qquad \int_\rho^\infty \frac{J_1(x)}{x^{2n+1}}\mathrm{d}x = \frac{1}{4n^2-1}\left[2n\frac{J_1(\rho)}{\rho^{2n}} + \frac{J_1'(\rho)}{\rho^{2n-1}} - \int_\rho^\infty \frac{J_1(x)}{x^{2n-1}}\mathrm{d}x\right].$$

Proof. First, upon integration by parts (using the asymptotic behavior at infinity of J_1 given by (A.6)), we have

$$(A.19) \qquad \int_\rho^\infty \frac{J_1(x)}{x^{2n+1}}\mathrm{d}x = \frac{1}{2n}\frac{J_1(\rho)}{\rho^{2n}} + \frac{1}{2n}\int_\rho^\infty \frac{J_1'(x)}{x^{2n}}\mathrm{d}x.$$

Then, if we employ (A.1) to express J_1' in terms of J_1 and J_1'', we obtain

$$\int_\rho^\infty \frac{J_1(x)}{x^{2n+1}}\mathrm{d}x = \frac{1}{2n}\frac{J_1(\rho)}{\rho^{2n}} + \frac{1}{2n}\int_\rho^\infty \frac{J_1(x)}{x^{2n+1}}\mathrm{d}x - \frac{1}{2n}\int_\rho^\infty \frac{J_1(x)}{x^{2n-1}}\mathrm{d}x - \frac{1}{2n}\int_\rho^\infty \frac{J_1''(x)}{x^{2n-1}}\mathrm{d}x,$$

and hence

$$(A.20) \qquad \int_\rho^\infty \frac{J_1''(x)}{x^{2n-1}}\mathrm{d}x = \frac{J_1(\rho)}{\rho^{2n}} - (2n-1)\int_\rho^\infty \frac{J_1(x)}{x^{2n+1}}\mathrm{d}x - \int_\rho^\infty \frac{J_1(x)}{x^{2n-1}}\mathrm{d}x.$$

On the other hand, returning to (A.19) and integrating it by parts again, we arrive at

$$(A.21) \qquad \int_\rho^\infty \frac{J_1(x)}{x^{2n+1}}\mathrm{d}x = \frac{1}{2n}\frac{J_1(\rho)}{\rho^{2n}} + \frac{1}{2n(2n-1)}\frac{J_1'(\rho)}{\rho^{2n-1}} + \frac{1}{2n(2n-1)}\int_\rho^\infty \frac{J_1''(x)}{x^{2n-1}}\mathrm{d}x$$

$$= \frac{1}{2n-1}\frac{J_1(\rho)}{\rho^{2n}} + \frac{1}{2n(2n-1)}\frac{J_1'(\rho)}{\rho^{2n-1}} - \frac{1}{2n}\int_\rho^\infty \frac{J_1(x)}{x^{2n+1}}\mathrm{d}x - \frac{1}{2n(2n-1)}\int_\rho^\infty \frac{J_1(x)}{x^{2n-1}}\mathrm{d}x.$$

Here, in the second line, we eliminated the intergral term involving J_1'' using (A.20).

Rearranging the terms in (A.21) (solving for the quantity on the left-hand side), we deduce (A.18). $\qquad \square$

Lemma A.2. *For $\rho > 0$, we have*

$$(A.22) \qquad \int_\rho^\infty \frac{J_1(x)}{x^3}\mathrm{d}x = \frac{J_0(\rho)}{3\rho} + \frac{J_1(\rho)}{3\rho^2} - \frac{1}{3} - \frac{J_1(\rho)}{3} + \frac{\rho J_0(\rho)}{3} - \frac{\pi\rho}{6}\left[J_0(\rho)H_1(\rho) - J_1(\rho)H_0(\rho)\right].$$

Proof. Note that using (A.8), we have

$$(A.23) \qquad \int_\rho^\infty \frac{J_1(x)}{x^3}\mathrm{d}x = \frac{1}{2}\int_\rho^\infty \frac{J_0(x)}{x^2}\mathrm{d}x + \frac{1}{2}\int_\rho^\infty \frac{J_2(x)}{x^2}\mathrm{d}x.$$

Let us start by transforming the first term in (A.23), namely,

$$(A.24) \qquad \int_\rho^\infty \frac{J_0(x)}{x^2}\mathrm{d}x = \frac{J_0(\rho)}{\rho} + \int_\rho^\infty \frac{J_0'(x)}{x}\mathrm{d}x$$

$$= \frac{J_0(\rho)}{\rho} + J_0'(\rho) - \int_\rho^\infty J_0(x)\,\mathrm{d}x.$$

Here, in the first line, we employed integration by parts (together with the asymptotic behavior at infinity of J_0 given by (A.6)). To arrive at the second line, we used the identity

$$\frac{J_0'(x)}{x} = -J_0(x) - J_0''(x), \quad x \neq 0,$$

implied by (A.1).

Performing the same procedure with the second term in (A.23), we have

$$\int_\rho^\infty \frac{J_2(x)}{x^2}\mathrm{d}x = \frac{J_2(\rho)}{\rho} + \int_\rho^\infty \frac{J_2'(x)}{x}\mathrm{d}x$$

$$= \frac{J_2(\rho)}{\rho} + J_2'(\rho) - \int_\rho^\infty J_2(x)\,\mathrm{d}x + 4\int_\rho^\infty \frac{J_2(x)}{x^2}\mathrm{d}x,$$

and, hence,

$$(A.25) \qquad \int_\rho^\infty \frac{J_2(x)}{x^2}\mathrm{d}x = -\frac{1}{3}\left[\frac{J_2(\rho)}{\rho} + J_2'(\rho) - \int_\rho^\infty J_2(x)\,\mathrm{d}x\right].$$

Expressions (A.24) and (A.25) imply that the integral on the left-hand side of (A.23) is expressible in terms of two integral quantities: $\int_\rho^\infty J_0(x)\,dx$ and $\int_\rho^\infty J_2(x)\,dx$. Let us now show that these quantities are simply related:

$$(A.26) \qquad \int_\rho^\infty J_2(x)\,dx = \int_\rho^\infty J_0(x)\,dx + 2 J_1(\rho),$$

and, moreover,

$$(A.27) \qquad \int_\rho^\infty J_0(x)\,dx = 1 - \rho J_0(\rho) - \frac{\pi}{2}\rho\left[J_1(\rho) H_0(\rho) - J_0(\rho) H_1(\rho)\right].$$

Recalling asymptotics (A.6), we note that the integrals on the left-hand sides of (A.26), (A.27) are not absolutely convergent, and hence an additional care with technical manipulations is needed. Namely, we shall first replace the integration range (ρ, ∞) with (ρ, R) for arbitrary finite $R > \rho$, and then pass to the limit as $R \to +\infty$. We shall proceed in several steps.

Step 1: Establishing (A.26)

We start with (A.26) and use integral representation (A.3):

$$J_2(x) = \frac{1}{\pi}\int_0^\pi \cos(2t - x\sin t)\,dt = \frac{1}{\pi}\int_0^\pi \cos(x\sin t - 2t)\,dt.$$

Consequently, exchanging the order of integration (permissible due to the regularity of the integrand and finiteness of the integration limits), we obtain

$$(A.28) \qquad \int_\rho^R J_2(x)\,dx = \frac{1}{\pi}\int_0^\pi \frac{\sin(R\sin t - 2t) - \sin(\rho\sin t - 2t)}{\sin t}\,dt$$

$$= \frac{1}{\pi}\int_0^\pi \frac{\sin(R\sin t) - \sin(\rho\sin t)}{\sin t}\,dt - \frac{2}{\pi}\int_0^\pi \left[\sin(R\sin t)\sin t + \cos(R\sin t)\cos t\right]dt$$

$$+ \frac{2}{\pi}\int_0^\pi \left[\sin(\rho\sin t)\sin t + \cos(\rho\sin t)\cos t\right]dt$$

$$= \int_\rho^R J_0(x)\,dx - 2\left(J_1(R) - J_1(\rho)\right),$$

where we used the identities

$$\sin(R\sin t - 2t) = \left(1 - 2\sin^2 t\right)\sin(R\sin t) - 2\sin t \cos t \cos(R\sin t),$$

$$\sin(R\sin t)\sin t + \cos(R\sin t)\cos t = \cos(R\sin t - t),$$

$$\frac{1}{\pi}\int_0^\pi \cos(R\sin t - t)\,dt = J_1(R),$$

$$\frac{1}{\pi}\int_0^\pi \frac{\sin(R\sin t) - \sin(\rho\sin t)}{\sin t}\,dt = \int_\rho^R J_0(x)\,dx.$$

and, except for the last one, also their analogs with ρ instead of R. The first two of these identities are purely trigonometrical whereas the last two are due to (A.4)–(A.5).

Passing to the limit $R \to +\infty$ in (A.28) using asymptotics (A.6), we thus conclude with (A.26).

Step 2: Establishing (A.27)

To deduce relation (A.27), we use [19, (10.22.2)] to write

(A.29)
$$\int_\rho^R J_0(x)\,dx = \frac{\pi}{2}R\left[J_0(R) H_{-1}(R) - J_{-1}(R) H_0(R)\right] - \frac{\pi}{2}\rho\left[J_0(\rho) H_{-1}(\rho) - J_{-1}(\rho) H_0(\rho)\right]$$

$$= R J_0(R) - \rho J_0(\rho) + \frac{\pi}{2}R\left[J_1(R) H_0(R) - J_0(R) H_1(R)\right] - \frac{\pi}{2}\rho\left[J_1(\rho) H_0(\rho) - J_0(\rho) H_1(\rho)\right],$$

where we used (A.7), (A.17) in passing to the second line.

Note that, employing asymptotics (A.6), (A.15)–(A.16), we have, for $R \gg 1$,

$$RJ_0(R) + \frac{\pi}{2}R\left[J_1(R)H_0(R) - J_0(R)H_1(R)\right] = \cos\left(R - \frac{3\pi}{4}\right)\sin\left(R - \frac{\pi}{4}\right) - \cos\left(R - \frac{\pi}{4}\right)\sin\left(R - \frac{3\pi}{4}\right) + \mathcal{O}\left(\frac{1}{R^{1/2}}\right)$$

$$= 1 + \mathcal{O}\left(\frac{1}{R^{1/2}}\right).$$

Therefore, by passing to the limit as $R \to +\infty$ in (A.29), we obtain (A.27).

  Step 3: Conclusion of the proof

  Plugging (A.27) into (A.26) and (A.24), we obtain

$$\text{(A.30)} \qquad \int_\rho^\infty J_2(x)\,\mathrm{d}x = 1 + 2J_1(\rho) - \rho J_0(\rho) - \frac{\pi}{2}\rho\left[J_1(\rho)H_0(\rho) - J_0(\rho)H_1(\rho)\right],$$

$$\text{(A.31)} \qquad \int_\rho^\infty \frac{J_0(x)}{x^2}\mathrm{d}x = \frac{J_0(\rho)}{\rho} + J_0'(\rho) - 1 + \rho J_0(\rho) + \frac{\pi}{2}\rho\left[J_1(\rho)H_0(\rho) - J_0(\rho)H_1(\rho)\right],$$

respectively.

Substitution of (A.30) into (A.25) gives

$$\text{(A.32)} \qquad \int_\rho^\infty \frac{J_2(x)}{x^2}\mathrm{d}x = -\frac{J_2(\rho)}{3\rho} - \frac{1}{3}J_2'(\rho) + \frac{1}{3} + \frac{2}{3}J_1(\rho) - \frac{1}{3}\rho J_0(\rho) - \frac{\pi}{6}\rho\left[J_1(\rho)H_0(\rho) - J_0(\rho)H_1(\rho)\right].$$

Finally, using (A.31)–(A.32) in (A.23), we arrive at

$$\text{(A.33)} \quad \int_\rho^\infty \frac{J_1(x)}{x^3}\mathrm{d}x = \frac{J_0(\rho)}{2\rho} - \frac{J_2(\rho)}{6\rho} + \frac{1}{2}J_0'(\rho) - \frac{1}{6}J_2'(x) - \frac{1}{3} + \frac{\rho}{3}J_0(\rho) + \frac{1}{3}J_1(\rho) + \frac{\pi}{6}\rho\left[J_1(\rho)H_0(\rho) - J_0(\rho)H_1(\rho)\right].$$

Using (A.10) and the recursive identities (due to (A.8)–(A.9))

$$J_2(\rho) = \frac{2}{\rho}J_1(\rho) - J_0(\rho), \qquad J_3(\rho) = \frac{4}{\rho}J_2(\rho) - J_1(\rho) = \frac{8}{\rho^2}J_1(\rho) - \frac{4}{\rho}J_0(\rho) - J_1(\rho),$$

$$J_2'(\rho) = \frac{1}{2}\left[J_1(\rho) - J_3(\rho)\right] = -\frac{4}{\rho^2}J_1(\rho) + \frac{2}{\rho}J_0(\rho) + J_1(\rho),$$

we transform (A.33) into the desired relation (A.22). $\qquad\square$

Lemma A.3. *For $\rho > 0$, in addition to (A.22), we have the following identities*

$$\text{(A.34)} \qquad \int_\rho^\infty \frac{J_1(x)}{x}\mathrm{d}x = \frac{J_1(\rho)}{\rho^2} + \frac{J_1'(\rho)}{\rho} - \frac{1}{\rho}J_0(\rho) + 1 + J_1(\rho) - \rho J_0(\rho)$$

$$+ \frac{\pi\rho}{2}\left[J_0(\rho)H_1(\rho) - J_1(\rho)H_0(\rho)\right],$$

$$\text{(A.35)} \qquad \int_\rho^\infty \frac{J_1(x)}{x^5}\mathrm{d}x = \frac{4J_1(\rho)}{15\rho^4} + \frac{J_1'(\rho)}{15\rho^3} - \frac{J_0(\rho)}{45\rho} - \frac{J_1(\rho)}{45\rho^2} + \frac{1}{45} + \frac{J_1(\rho)}{45}$$

$$- \frac{\rho J_0(\rho)}{45} + \frac{\pi\rho}{90}\left[J_0(\rho)H_1(\rho) - J_1(\rho)H_0(\rho)\right],$$

$$\text{(A.36)} \qquad \int_\rho^\infty \frac{J_1(x)}{x^7}\mathrm{d}x = \frac{6}{35}\frac{J_1(\rho)}{\rho^6} + \frac{1}{35}\frac{J_1'(\rho)}{\rho^5} - \frac{4J_1(\rho)}{525\rho^4} - \frac{J_1'(\rho)}{525\rho^3} + \frac{J_0(\rho)}{1575\rho} + \frac{J_1(\rho)}{1575\rho^2}$$

$$- \frac{1}{1575} - \frac{J_1(\rho)}{1575} + \frac{\rho J_0(\rho)}{1575} - \frac{\pi\rho}{3150}\left[J_0(\rho)H_1(\rho) - J_1(\rho)H_0(\rho)\right].$$

Proof. We shall prove identities (A.34)–(A.36) sequentially.

  Applying Lemma A.1 with $n = 1$, we have

$$\int_\rho^\infty \frac{J_1(x)}{x^3}\mathrm{d}x = \frac{2}{3}\frac{J_1(\rho)}{\rho^2} + \frac{1}{3}\frac{J_1'(\rho)}{\rho} - \frac{1}{3}\int_\rho^\infty \frac{J_1(x)}{x}\mathrm{d}x,$$

and, hence,

$$\text{(A.37)} \qquad \int_\rho^\infty \frac{J_1(x)}{x}\mathrm{d}x = 2\frac{J_1(\rho)}{\rho^2} + \frac{J_1'(\rho)}{\rho} - 3\int_\rho^\infty \frac{J_1(x)}{x^3}\mathrm{d}x.$$

Using Lemma A.2, (A.37) becomes (A.34).

Application of Lemma A.1 with $n = 2$ yields

$$(A.38) \qquad \int_\rho^\infty \frac{J_1(x)}{x^5} dx = \frac{4}{15} \frac{J_1(\rho)}{\rho^4} + \frac{1}{15} \frac{J_1'(\rho)}{\rho^3} - \frac{1}{15} \int_\rho^\infty \frac{J_1(x)}{x^3} dx.$$

With the help of Lemma A.2, (A.38) transforms into (A.35).

Finally, we use Lemma A.1 with $n = 3$ to obtain

$$(A.39) \qquad \int_\rho^\infty \frac{J_1(x)}{x^7} dx = \frac{6}{35} \frac{J_1(\rho)}{\rho^6} + \frac{1}{35} \frac{J_1'(\rho)}{\rho^5} - \frac{1}{35} \int_\rho^\infty \frac{J_1(x)}{x^5} dx.$$

Substitution of the already proven identity (A.35) into (A.39) furnishes (A.36). □

Lemma A.4. *For $\alpha \in \mathbb{R}$, m, $n \in \mathbb{N}_0$, we have the following identities*

$$(A.40) \qquad \int_0^{2\pi} \cos(\alpha \cos\theta) \cos^{2m+1}\theta \, \sin^n \theta d\theta = 0,$$

$$(A.41) \qquad \int_0^{2\pi} \cos(\alpha \cos\theta) \cos^m \theta \, \sin^{2n+1}\theta d\theta = 0,$$

$$(A.42) \qquad \int_0^{2\pi} \sin(\alpha \cos\theta) \cos^m \theta \, \sin^{2n+1}\theta d\theta = 0,$$

$$(A.43) \qquad \int_0^{2\pi} \sin(\alpha \cos\theta) \cos^{2m}\theta \, \sin^n \theta d\theta = 0.$$

Proof. We shall prove identities (A.40)–(A.43) one after another.

Using periodicity of the integrand, we can shift the interval from $(0, 2\pi)$ to $(-\pi/2, 3\pi/2)$ and further split it in two:

$$(A.44) \qquad \int_0^{2\pi} \cos(\alpha \cos\theta) \cos^{2m+1}\theta \, \sin^n \theta d\theta = \int_{-\pi/2}^{3\pi/2} \cos(\alpha \cos\theta) \cos^{2m+1}\theta \, \sin^n \theta d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \ldots + \int_{\pi/2}^{3\pi/2} \ldots .$$

Performing the change of variable $y = \sin\theta$, $dy = \cos\theta d\theta$ in each of the integrals on the second line of (A.44), we have

$$\int_{-\pi/2}^{\pi/2} \cos(\alpha \cos\theta) \cos^{2m+1}\theta \, \sin^n \theta d\theta = \int_{-1}^1 \cos\left(\alpha\sqrt{1-y^2}\right) \left(1-y^2\right)^m y^n dy,$$

$$\int_{\pi/2}^{3\pi/2} \cos(\alpha \cos\theta) \cos^{2m+1}\theta \, \sin^n \theta d\theta = \int_1^{-1} \cos\left(-\alpha\sqrt{1-y^2}\right) \left(-\sqrt{1-y^2}\right)^{2m} y^n dy$$

$$= -\int_{-1}^1 \cos\left(\alpha\sqrt{1-y^2}\right) \left(1-y^2\right)^m y^n dy.$$

Here, we used the fact that $\cos\theta = \sqrt{1-\sin^2\theta} > 0$ for $\theta \in (-\pi/2, \pi/2)$ and $\cos\theta = -\sqrt{1-\sin^2\theta} < 0$ for $\theta \in (\pi/2, 3\pi/2)$ which makes both integral quantities on the second line of (A.44) opposite to each other in sign and thus entails (A.40).

To show (A.41), a useful change of variable is $y = \cos\theta$, $dy = -\sin\theta d\theta$. This, upon splitting the integration range to into $(0, \pi)$ and $(\pi, 2\pi)$, leads to

$$\int_0^{2\pi} \cos(\alpha \cos\theta) \cos^m \theta \, \sin^{2n+1}\theta d\theta = -\int_1^{-1} \cos(\alpha y) y^m \left(1-y^2\right)^n dy - \int_{-1}^1 \cos(\alpha y) y^m \left(1-y^2\right)^n dy$$

$$= \int_{-1}^1 \cos(\alpha y) y^m \left(1-y^2\right)^n dy - \int_{-1}^1 \cos(\alpha y) y^m \left(1-y^2\right)^n dy$$

$$= 0.$$

Similarly, we have

$$\int_0^{2\pi} \sin(\alpha\cos\theta)\cos^m\theta \, \sin^{2n+1}\theta\mathrm{d}\theta = -\int_1^{-1} \sin(\alpha y)\, y^m \left(1-y^2\right)^n \mathrm{d}y - \int_{-1}^1 \sin(\alpha y)\, y^m \left(1-y^2\right)^n \mathrm{d}y$$
$$= 0,$$

which proves (A.42).

Finally, to show (A.43), we use the fact that identity (A.40) holds, in particular, for any $\alpha \in [0, \beta]$ with arbitrary $\beta \geq 0$. Integrating it in α over this interval and interchanging the order of integration (permissible by the regularity of the integrand and the finite integration range), we obtain

$$0 = \int_0^\beta \int_0^{2\pi} \cos(\alpha\cos\theta)\cos^{2m+1}\theta \, \sin^n\theta\mathrm{d}\theta\mathrm{d}\alpha = \int_0^{2\pi} \sin(\beta\cos\theta)\cos^{2m}\theta \, \sin^n\theta\mathrm{d}\theta.$$

Since the same reasoning also works $\beta \leq 0$ by working with an interval $[\beta, 0]$, identity (A.43) is thus proved up to a change of the notation β to α. $\qquad\square$

References

[1] Andersen, F. H., Pedersen, L. B., "Some relations between potential fields and the strength and center of their sources", Geophys. Prospect., 27 (4), 761–774, 1979.

[2] Baratchart, L., Chevillard, S., Hardin, D., Leblond, J., Lima, E., Marmorat, J. P., "Magnetic moment estimation and bounded extremal problems", Inverse Problems and Imaging, 13 (1), 2019.

[3] Baratchart, L., Chevillard, S., Leblond, J., Lima, E., Ponomarev, D., "Asymptotic method for estimating magnetic moments from field measurements on a planar grid" (preprint), HAL Id: hal-01421157, 2018.

[4] Baratchart, L., Hardin, D. P., Lima, E. A., Saff, E. B., Weiss, B. P., "Characterizing kernels of operators related to thin-plate magnetizations via generalizations of Hodge decompositions", Inverse Problems, 29 (1), 2013.

[5] Baratchart, L., Leblond, J., Lima, E., Ponomarev, D., "Magnetization moment recovery using Kelvin transformation and Fourier analysis", J. Phys. Conf. Ser., 904, 2017.

[6] Baratchart, L., Villalobos-Guillén, C., Hardin, D. P., Northington, M. C., Saff, E. B., "Inverse potential problems for divergence of measures with total variation regularization", Foundations of Computational Mathematics, 20 (5), 1273–1307, 2020.

[7] Baratchart, L., Leblond, J., Nemaire, M., "Silent sources in L^p and Helmholtz-type decompositions", HAL preprint hal:03915548, 2023.

[8] Blakely, R.J., "Potential theory in gravity and magnetic applications", Cambridge University Press, 1996.

[9] Caratori Tontini, F., Pedersen, L. B., "Interpreting magnetic data by integral moments", Geophys. J. Int. 174 (3), 815–824, 2008.

[10] Fu, R. R., Lima, E. A., Volk, M. W., Trubko, R., "High-sensitivity moment magnetometry with the quantum diamond microscope", Geochemistry, Geophysics, Geosystems, 21 (8), 2020.

[11] Gunn, P. J., "Linear transformations of gravity and magnetic fields", Geophys. Prospect., 23 (2), 300–312, 1975.

[12] Helbig, K., "Some integrals of magnetic anomalies and their relation to the parameters of the disturbing body", Zeitschrift für Geophysik, 29(2), 83–96, 1963.

[13] Lima, E. A., Weiss, B. P., Baratchart, L., Hardin, D. P., Saff, E. B., "Fast inversion of magnetic field maps of unidirectional planar geological magnetization", J. Geophys. Res.: Solid Earth, 118 (6), 2723–2752, 2013.

[14] Lima, E. A., Weiss, B. P., "Obtaining vector magnetic field maps from single-component measurements of geological samples", Journal of Geophysical Research: Solid Earth, 114 (B6), 2009.

[15] Lima, E. A., Weiss, B. P., "Ultra-high sensitivity moment magnetometry of geological samples using magnetic microscopy", Geochemistry, Geophysics, Geosystems, 17 (9), 3754–3774, 2016.

[16] Lima, E. A., Weiss, B. P., Borlina, C. S., Baratchart, L., Hardin, D. P., "Estimating the net magnetic moment of geological samples from planar field maps using multipoles", Geochemistry, Geophysics, Geosystems, 24, 2023.

[17] McKenzie, B., Foss, C., Hillan, D., "Issues related to determination of the horizontal centre of magnetization", ASEG Extended Abstracts, 2012(1), 1–4, 2012.

[18] Medeiros, W. E., Silva, J. B. C., "Simultaneous estimation of total magnetization direction and 3-D spatial orientation", Geophys., 60 (5), 1365–1367, 1995.

[19] Olver, F. W. J., Lozier, D. W., Boisvert, R. F., Clark, C. W., "NIST Handbook of Mathematical Functions", Cambridge University Press, 2010.

[20] Phillips, J. D., "Can we estimate total magnetization directions from aeromagnetic data using Helbig's integrals?", Earth, Planets and Space, 57, 681–689, 2005.

[21] Ponomarev, D., "Some inverse problems with partial data", Doctoral thesis, Université de Nice - Sophia Antipolis (University of Côte d'Azur), 2016.

[22] Ponomarev, D., "A method to extrapolate the data for the inverse magnetisation problem with a planar sample", Inverse Problems: Modeling and Simulation (IPMS 2024) - Extended abstracts (to appear), 2024.

[23] Schmidt, P.W., Clark, D.A., "The calculation of magnetic components and moments from TMI: A case study from the Tuckers igneous complex, Queensland", Exploration Geophysics, 29(4), 609–614, 1998.

[24] Stein, E. M., Weiss, G., "Introduction to Fourier Analysis on Euclidean Spaces", Princeton University Press, 2016.

[25] Strichartz, R., "Guide to Distribution Theory and Fourier Transforms", CRC Press, 1994.

[26] Weiss, B. P., Lima, E. A., Fong, L. E., Baudenbacher, F. J., "Paleomagnetic analysis using SQUID microscopy", Journal of Geophysical Research: Solid Earth, 112 (B9), 2007.