A short note on the appearance of the simplest antilinear ODE in several physical contexts

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Abstract

In this short note, we review several one-dimensional problems such as those involving linear Schrödinger equation, variable-coefficient Helmholtz equation, Zakharov-Shabat system and Kubelka-Munk equations. We show that they all can be reduced to solving one simple antilinear ordinary differential equation $u'(x) = f(x)\overline{u(x)}$ or its nonhomogeneous version $u'(x) = f(x)\overline{u(x)} + g(x)$, $x \in (0, x_0) \subset \mathbb{R}$. We point out some of the advantages of the proposed reformulation and call for further investigation of the obtained ODE.

1   Introduction

Many physical phenomena can be directly described by or reduced to systems of differential equations having certain structural properties. Restricting ourselves here to linear one-dimensional settings, we are concerned with a pair of first-order ODEs whose matrix is antidiagonal with complex-conjugate elements. Namely, given $x_0 \in \mathbb{R}$ and complex-valued function $f(x)$, we consider the equation

$$U'(x) = \begin{pmatrix} 0 & f(x) \\ \overline{f(x)} & 0 \end{pmatrix} U(x), \quad x \in (0, x_0), \tag{1}$$

as well as its nonhomogeneous analog

$$U'(x) = \begin{pmatrix} 0 & f(x) \\ \overline{f(x)} & 0 \end{pmatrix} U(x) + G(x), \quad x \in (0, x_0), \tag{2}$$

where $U(x) \equiv (u_1(x), u_2(x))^T \in \mathbb{C}^2$ is an unknown solution-vector, $G(x) \equiv (g_1(x), g_2(x))^T \in \mathbb{C}^2$ is a given vector-function, and each of equations (1)–(2) is supplemented by the initial condition $U(0) = U_0 \in \mathbb{C}^2$. Here and onwards, we employ the notation $\overline{\cdot}$ to denote complex conjugation.

Similarly to Hamiltonian, Dirac and more general canonical systems (see e.g. [12]), equations (1)–(2) constitute an important class of dynamical systems for two reasons. On the one hand, as we shall further see, formulations of several important problems are reducible to either (1) or (2). On the other hand, these systems are close to being exactly solvable in the following sense. Let us focus on (1) and consider the more general system

$$U'(x) = \begin{pmatrix} p(x) & r(x) \\ s(x) & q(x) \end{pmatrix} U(x), \quad x \in (0, x_0). \tag{3}$$

We note that the diagonal elements in the matrix of (3) can be removed by the exponential multiplier transform.

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Namely, by setting

$$V(x) := \begin{pmatrix} e^{-\int_0^x p(\tau)\,d\tau} & 0 \\ 0 & e^{-\int_0^x q(\tau)\,d\tau} \end{pmatrix} U(x),$$

one can observe that $V(x)$ satisfies

$$V'(x) = \begin{pmatrix} 0 & r(x)\exp\left(-\int_0^x [p(\tau)-q(\tau)]\,d\tau\right) \\ s(x)\exp\left(\int_0^x [p(\tau)-q(\tau)]\,d\tau\right) & 0 \end{pmatrix} V(x), \quad x \in (0, x_0), \qquad (4)$$

with the initial condition $V(0) = U_0$. Now, if the anti-diagonal elements of the matrix in the right-hand side of (4) are equal, i.e.

$$r(x)\exp\left(-\int_0^x [p(\tau)-q(\tau)]\,d\tau\right) = s(x)\exp\left(\int_0^x [p(\tau)-q(\tau)]\,d\tau\right) =: c_1(x), \qquad (5)$$

then the solution can be written explicitly as

$$V(x) = \left[\cosh\left(\int_0^x c_1(\tau)\,d\tau\right) I + \sinh\left(\int_0^x c_1(\tau)\,d\tau\right) S\right] U_0,$$

where

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \qquad (6)$$

However, condition (5), which amounts to the assumption

$$s(x) = r(x)\exp\left(-2\int_0^x [p(\tau)-q(\tau)]\,d\tau\right), \qquad (7)$$

may be too restrictive. Indeed, in view of multiple possible similarity transformations allowing to rewrite (3) in different equivalent forms, we want to have a clearly identifiable matrix structure which should be, on the one hand, immediately recognisable and, on the other hand, leading to a solution simplification or even an explicit solution. Such an identifiable structure may be, for example, a pairwise relation between some of the elements of the matrix in (3). The explicit solvability condition, nevertheless, plays against any visible structural property of the matrix: even though assumption (7) leads to a closed-form solution, it implies a rather complicated relation between the matrix elements. Condition (7) is very specific and thus unlikely to be satisfied for any easily describable class of matrix elements unless, of course, $p \equiv q$, which would then also entail that $r \equiv s$. This exactly solvable case with equal diagonal and anti-diagonal elements in (3) may be sometimes valuable but it does not seem to be the one that covers many fruitful applications.

It turns out that condition (7) has an analog which is less stringent in form of the matrix elements in (3), with more pertinence to important physical contexts, and, at the same time, it still leads to a significant simplification of the solution procedure (and, at least in some cases, also to closed-form solutions). This condition reads

$$s(x) = \overline{r(x)}\exp\left(-2\,\mathrm{Re}\int_0^x [p(\tau)-q(\tau)]\,d\tau\right). \qquad (8)$$

Despite the similarity to (7), condition (8) is easier to satisfy while preserving a visible matrix structure. Indeed, if $p(x) - q(x)$ is a purely imaginary function (e.g., in particular, when $p \equiv \bar{q}$), the complicated exponential factor in (8) disappears. In this case, the implied condition $s(x) = \overline{r(x)}$ is clearly indentifiable but far from being trivial since, as we shall see, it covers a variety of different practical applications. This reasoning motivates us to consider (1) as well as its nonhogeneous analog (2).

The plan of this note is as follows. Section 2 is dedicated to the transformation of (1) and (2) into the homogeneous antilinear ODE

$$u'(x) = f(x)\overline{u(x)}, \quad x \in (0, x_0),$$ (9)

and its nonhomogeneous analog

$$u'(x) = f(x)\overline{u(x)} + g(x), \quad x \in (0, x_0),$$ (10)

respectively. Next, in Section 3, we outline some relevant physical applications, i.e. problems which, upon appropriate transformations, can be recast as (1) or (2) and are thus reducible to formulations involving antilinear ODE (9), or, in one case, its nonhomogeneous version (10). Finally, in Section 4, we conclude with some remarks on how antilinear ODEs can be constructively addressed further and briefly mention a couple of other applications.

2 Transformation of an antidiagonal problem into an antilinear ODE

2.1 Homogeneous case: from (1) to (9)

We consider (1) supplemented with the initial data $U(0) = U_0 \equiv (u_1^0, u_2^0)^T$, where $u_1^0, u_2^0 \in \mathbb{C}$ are arbitrary constants, and we devise a transformation that allows construction of the solution of system (1) in terms of the solution of an antilinear ODE of the form (9).

Let us first motivate our approach to construction of such a transformation. To this effect, given a complex-valued function $f(x)$, it is instructive to consider the elementary differential equation for $v(x)$

$$v'(x) = f(x)v(x), \quad x \in (0, x_0),$$ (11)

with the initial condition $v(0) = 1$. On the one hand, (11) is in separable form and hence can be integrated directly to yield the solution

$$v(x) = \exp\left(\int_0^x f(\tau)\,d\tau\right).$$ (12)

On the other hand, rewriting (11) in the integral form

$$v(x) = 1 + \int_0^x f(\tau)v(\tau)\,d\tau, \quad x \in (0, x_0),$$

the Picard iterative process gives

$$v(x) = 1 + \int_0^x f(\tau)\,d\tau + \int_0^x f(\tau_2)\int_0^{\tau_2} f(\tau_1)\,d\tau_1 d\tau_2 + \int_0^x f(\tau_3)\int_0^{\tau_3} f(\tau_2)\int_0^{\tau_2} f(\tau_1)\,d\tau_1 d\tau_2 d\tau_3 + \dots.$$ (13)

Comparison of (12) with (13) results in important identities

$$\exp\left(\int_0^x f(\tau)\,d\tau\right) = 1 + \int_0^x f(\tau)\,d\tau + \int_0^x f(\tau_2)\int_0^{\tau_2} f(\tau_1)\,d\tau_1 d\tau_2 + \int_0^x f(\tau_3)\int_0^{\tau_3} f(\tau_2)\int_0^{\tau_2} f(\tau_1)\,d\tau_1 d\tau_2 d\tau_3 + \dots,$$

$$\sinh\left(\int_0^x f(\tau)\,d\tau\right) = \int_0^x f(\tau)\,d\tau + \int_0^x f(\tau_3)\int_0^{\tau_3} f(\tau_2)\int_0^{\tau_2} f(\tau_1)\,d\tau_1 d\tau_2 d\tau_3 + \dots,$$ (14)

$$\cosh\left(\int_0^x f(\tau)\,d\tau\right) = 1 + \int_0^x f(\tau_2)\int_0^{\tau_2} f(\tau_1)\,d\tau_1 d\tau_2 + \int_0^x f(\tau_4)\int_0^{\tau_4} f(\tau_3)\int_0^{\tau_3} f(\tau_2)\int_0^{\tau_2} f(\tau_1)\,d\tau_1 d\tau_2 d\tau_3 d\tau_4 + \dots,$$ (15)

where we used the identity $\exp(z) = \cosh(z) + \sinh(z)$, $z \in \mathbb{C}$, and the parity argument to split the terms: sinh is an odd function and hence (14) may contain only odd number of multiplicative instances of f, and similarly, (15)

may contain only terms with even number of multiplications by f due to cosh being an even function.

Now, similarly to (14)–(15), let us consider the following quantities

$$S_f(x) := \int_0^x f(\tau_1)\,d\tau_1 + \int_0^x f(\tau_3)\int_0^{\tau_3}\overline{f(\tau_2)}\int_0^{\tau_2} f(\tau_1)\,d\tau_1 d\tau_2 d\tau_3 + \dots, \tag{16}$$

$$C_f(x) := 1 + \int_0^x f(\tau_2)\int_0^{\tau_2}\overline{f(\tau_1)}d\tau_1 d\tau_2 + \int_0^x f(\tau_4)\int_0^{\tau_4}\overline{f(\tau_3)}\int_0^{\tau_3} f(\tau_2)\int_0^{\tau_2}\overline{f(\tau_1)}d\tau_1 d\tau_2 d\tau_3 d\tau_4 + \dots. \tag{17}$$

Let us show that (16)–(17) are inherent to an algebraic structure underlying (1). To this effect, we rewrite (1) in the integral form

$$U(x) = U_0 + \int_0^x A(\tau)U(\tau)\,d\tau, \quad x \in (0, x_0), \qquad A(\tau) := \begin{pmatrix} 0 & f(\tau) \\ \overline{f(\tau)} & 0 \end{pmatrix}, \tag{18}$$

and note that

$$A(\tau_2)A(\tau_1) = \begin{pmatrix} f(\tau_2)\overline{f(\tau_1)} & 0 \\ 0 & \overline{f(\tau_2)}f(\tau_1) \end{pmatrix}, \quad A(\tau_3)A(\tau_2)A(\tau_1) = \begin{pmatrix} 0 & f(\tau_3)\overline{f(\tau_2)}f(\tau_1) \\ \overline{f(\tau_3)}f(\tau_2)\overline{f(\tau_1)} & 0 \end{pmatrix},$$

$$A(\tau_4)A(\tau_3)A(\tau_2)A(\tau_1) = \begin{pmatrix} f(\tau_4)\overline{f(\tau_3)}f(\tau_2)\overline{f(\tau_1)} & 0 \\ 0 & \overline{f(\tau_4)}f(\tau_3)\overline{f(\tau_2)}f(\tau_1) \end{pmatrix}, \quad \dots.$$

Therefore, writing out Picard iterations for solving (18), we obtain

$$U(x) = \begin{pmatrix} C_f(x) & S_f(x) \\ S_{\overline{f}}(x) & C_{\overline{f}}(x) \end{pmatrix} U_0 = \begin{pmatrix} C_f(x) & S_f(x) \\ \overline{S_f(x)} & \overline{C_f(x)} \end{pmatrix} U_0. \tag{19}$$

Furthermore, it is easy to see from (16)–(17) that $S_f(x)$, $C_f(x)$ obey the following intertwining relation

$$C_f'(x) = f(x)\overline{S_f(x)}, \qquad S_f'(x) = f(x)\overline{C_f(x)}, \qquad x \in (0, x_0), \tag{20}$$

and the conditions $S_f(0) = 0$, $C_f(0) = 1$. Introducing another pair of functions

$$Z_+(x) := C_f(x) + S_f(x), \qquad Z_-(x) := C_f(x) - S_f(x),$$

we decouple (20) as

$$Z_+'(x) = f(x)\overline{Z_+(x)}, \quad x \in (0, x_0), \qquad Z_+(0) = 1, \tag{21}$$

$$Z_-'(x) = -f(x)\overline{Z_-(x)}, \quad x \in (0, x_0), \qquad Z_-(0) = 1. \tag{22}$$

Equations (21)–(22) are two separate instances of the initial-value problem with the antilinear ODE given by (9). Solution of this ODE would thus yields the solutions of (21)–(22) and, consequently, also of (20), providing $S_f(x)$, $C_f(x)$ appearing in (19) which furnishes the solution of (1).

2.2 Nonhomogeneous case: from (2) to (10)

Let us now consider (2) with $G(x) \equiv (g_1(x), g_2(x))^T$ and subject to the initial condition $U(0) = U_0 \equiv (u_1^0, u_2^0)^T$. We are going to show that, in particular case where

$$g_2(x) = i\,\overline{g_1(x)}, \qquad u_2^0 = i\,\overline{u_1^0}, \tag{23}$$

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the solution of (2) can be constucted in terms of solutions of two instances of problem (10). As we shall see in Subsection 3.2, assumption (23) will be satisfied in at least one important practical context.

Similarly to (16)–(17), let us introduce

$$S_{f,h}(x) := \int_0^x f(\tau_1) h(\tau_1) \, d\tau_1 + \int_0^x f(\tau_3) \int_0^{\tau_3} \overline{f(\tau_2)} \int_0^{\tau_2} f(\tau_1) h(\tau_1) \, d\tau_1 d\tau_2 d\tau_3 + \dots, \tag{24}$$

$$C_{f,h}(x) := h(x) + \int_0^x f(\tau_2) \int_0^{\tau_2} \overline{f(\tau_1)} h(\tau_1) \, d\tau_1 d\tau_2 \tag{25}$$
$$+ \int_0^x f(\tau_4) \int_0^{\tau_4} \overline{f(\tau_3)} \int_0^{\tau_3} f(\tau_2) \int_0^{\tau_2} \overline{f(\tau_1)} h(\tau_1) \, d\tau_1 d\tau_2 d\tau_3 d\tau_4 + \dots.$$

Rewriting (2) in the integral form

$$U(x) = U_0 + \int_0^x G(\tau) \, d\tau + \int_0^x A(\tau) U(\tau) \, d\tau, \quad x \in (0, x_0), \qquad A(\tau) := \begin{pmatrix} 0 & f(\tau) \\ \overline{f(\tau)} & 0 \end{pmatrix}, \tag{26}$$

it is straightforward to see that Picard iterations give

$$U(x) = \begin{pmatrix} S_{f,h_2}(x) + C_{f,h_1}(x) \\ S_{\overline{f},h_1}(x) + C_{\overline{f},h_2}(x) \end{pmatrix},$$

where $S_{f,h}(x)$, $C_{f,h}(x)$ are as defined by (24)–(25), and

$$h_1(x) := u_1^0 + \int_0^x g_1(\tau) \, d\tau, \qquad h_2(x) := u_2^0 + \int_0^x g_2(\tau) \, d\tau. \tag{27}$$

By means of differentiation of $S_{f,h_2}(x)$ and $C_{f,h_1}(x)$, we obtain the following intertwining relation

$$C'_{f,h_1}(x) = h'_1(x) + f(x) S_{\overline{f},h_1}(x), \quad S'_{f,h_2}(x) = f(x) C_{\overline{f},h_2}(x), \qquad x \in (0, x_0), \tag{28}$$

which is to be supplemented by the conditions $C_{f,h_1}(0) = h_1(0)$, $S_{f,h_2}(0) = 0$.

Note that, from (24)–(25), $S_{\overline{f},h_1}(x) = \overline{S_{f,\overline{h_1}}(x)}$ and $C_{\overline{f},h_2}(x) = \overline{C_{f,\overline{h_2}}(x)}$. Moreover, assumption (23) entails that $h_2(x) = i\overline{h_1(x)}$, and by linearity in h of $S_{f,h}(x)$, $C_{f,h}(x)$, we have $S_{f,h_2}(x) = iS_{f,\overline{h_1}}(x)$ and $\overline{C_{f,\overline{h_2}}(x)} = i\overline{C_{f,h_1}(x)}$. Consequently, relation (28) becomes

$$C'_{f,h_1}(x) = h'_1(x) + f(x) \overline{S_{f,\overline{h_1}}(x)}, \quad S'_{f,\overline{h_1}}(x) = f(x) \overline{C_{f,h_1}(x)}, \qquad x \in (0, x_0). \tag{29}$$

Setting

$$Z_+(x) := C_{f,h_1}(x) + S_{f,\overline{h_1}}(x), \quad Z_-(x) := C_{f,h_1}(x) - S_{f,\overline{h_1}}(x), \tag{30}$$

we obtain from (29) two decoupled ODE problems

$$Z'_+(x) = f(x) \overline{Z_+(x)} + g_1(x), \qquad Z_+(0) = u_1^0, \tag{31}$$

$$Z'_-(x) = -f(x) \overline{Z_-(x)} + g_1(x), \qquad Z_-(0) = u_1^0, \tag{32}$$

each of them are of the form (10).

3 Some physical contexts leading to (1) and (2)

3.1 Linear Schrödinger equation

Consider the stationary linear Schrödinger equation in $1D$, with a potential $a(x) > 0$,

$$u''(x) + a(x)u(x) = 0, \quad x \in (0, x_0). \tag{33}$$

We focus here on the initial-value problem, i.e. we supplement (33) with the boundary conditions $u(0) = u_0$, $u'(0) = u_1$, but boundary-value problems on $(0, x_0)$, with x_0 being finite or infinite, could also be treated. We assume $a \in C^1([0, x_0])$.

Introducing the vector-function

$$U(x) := \begin{pmatrix} u(x) \\ \frac{1}{a^{1/2}(x)} u'(x) \end{pmatrix}, \tag{34}$$

we observe that $U(x)$ satisfies

$$U'(x) = \begin{pmatrix} u'(x) \\ -\frac{a'(x)}{2a^{3/2}(x)} u'(x) + \frac{1}{a^{1/2}(x)} u''(x) \end{pmatrix} = A(x)U(x), \tag{35}$$

with

$$A(x) := \begin{pmatrix} 0 & a^{1/2}(x) \\ -a^{1/2}(x) & -\frac{a'(x)}{2a(x)} \end{pmatrix},$$

and $U(0) = \left(u_0, u_1/a^{1/2}(0)\right)^T$. Here, in the second inequality of (35), we used (33) to eliminate $u''(x)$.

By writing,

$$A(x) = a^{1/2}(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{a'(x)}{2a(x)} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

we note that the first matrix in the right-hand side is diagonalisable as follows

$$P \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} P^{-1} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad P := \frac{1}{2^{1/2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}, \quad P^{-1} = \frac{1}{2^{1/2}} \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}. \tag{36}$$

Consequently, introducing $V(x) := PU(x)$, we multiply the both sides of (35) by P and thus transform it into

$$V'(x) = PA(x)P^{-1}V(x) = B(x)V(x), \quad x \in (0, x_0), \tag{37}$$

with

$$B(x) := a^{1/2}(x) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \frac{a'(x)}{4a(x)} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = \begin{pmatrix} b_0(x) & 0 \\ 0 & \overline{b_0(x)} \end{pmatrix} - \frac{a'(x)}{4a(x)} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$b_0(x) := ia^{1/2}(x) - \frac{a'(x)}{4a(x)}, \tag{38}$$

and supplemented by the initial condition

$$V(0) = PU(0) = \frac{1}{2^{1/2}} \begin{pmatrix} iu_0 + u_1/a^{1/2}(0) \\ u_0 + iu_1/a^{1/2}(0) \end{pmatrix}.$$

Furthermore, introducing

$$W(x) := \begin{pmatrix} \exp\left(-\int_0^x b_0(\tau)\,\mathrm{d}\tau\right) & 0 \\ 0 & \exp\left(-\int_0^x \overline{b_0(\tau)}\mathrm{d}\tau\right) \end{pmatrix} V(x),$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} \exp\left(-\int_0^x b_0(\tau)\,\mathrm{d}\tau\right) & 0 \\ 0 & \exp\left(-\int_0^x \overline{b_0(\tau)}\mathrm{d}\tau\right) \end{pmatrix} = \begin{pmatrix} -b_0(x)\exp\left(-\int_0^x b_0(\tau)\,\mathrm{d}\tau\right) & 0 \\ 0 & -\overline{b_0(x)}\exp\left(-\int_0^x \overline{b_0(\tau)}\mathrm{d}\tau\right) \end{pmatrix}.$$

Therefore, (37) entails

$$W'(x) = -\frac{a'(x)}{4a(x)}\begin{pmatrix} 0 & -i\exp\left(-\int_0^x \left[b_0(\tau)-\overline{b_0(\tau)}\right]\mathrm{d}\tau\right) \\ i\exp\left(\int_0^x \left[b_0(\tau)-\overline{b_0(\tau)}\right]\mathrm{d}\tau\right) & 0 \end{pmatrix} W(x),$$

which, recalling (38), we can rewrite as

$$W'(x) = C(x)W(x), \quad x \in (0, x_0), \tag{39}$$

with

$$C(x) := \begin{pmatrix} 0 & c_0(x) \\ c_0(x) & 0 \end{pmatrix}, \qquad c_0(x) := \frac{ia'(x)}{4a(x)}\exp\left(-2i\int_0^x a^{1/2}(\tau)\,\mathrm{d}\tau\right),$$

and the initial condition

$$W(0) = V(0) = \frac{1}{2^{1/2}}\begin{pmatrix} iu_0 + u_1/a^{1/2}(0) \\ u_0 + iu_1/a^{1/2}(0) \end{pmatrix}.$$

The steps described above draw from [10] (see also [4]) and provide one way to rewrite the linear Schrödinger equation in the form (1), but this approach is not the only one. Alternative reduction procedures may be more cumbersome but more beneficial in practice, depending on a final goal. For instance, in [2], the initial vectorisation of (33) is different from (34) yet other steps of the transformation are ideologically similar.

3.2   Helmholtz equation

Stationary problems for the wave propagation in heterogeneous media are described by the Helmholtz equation whose $1D$ version is given by

$$(\alpha(x)u'(x))' + \beta(x)u(x) = f(x), \quad x \in (0, x_0). \tag{40}$$

Here, $\alpha(x)$, $\beta(x) > 0$ are material parameters and $f(x)$ is the source term. As in Subsection 3.1, we suppose that (40) is supplemented by the initial conditions $u(0) = u_0$, $u'(0) = u_1$. Furthermore, we assume that $f(x)$, u_0, u_1 are all real-valued. This assumption does not reduce generality since (40) is linear with real-valued $\alpha(x)$, $\beta(x)$ and hence a real-valued problem can be solved separately for real and imaginary parts of the solution of the original equation.

Setting

$$U(x) := \begin{pmatrix} \alpha^{1/2}(x)\beta^{1/2}(x)u(x) \\ \alpha(x)u'(x) \end{pmatrix},$$

we recast (40) in the vector form

$$U'(x) = A(x)U(x) + F_0(x), \quad x \in (0, x_0),$$ (41)

with

$$A(x) := \begin{pmatrix} \frac{[\alpha(x)\beta(x)]'}{2\alpha(x)\beta(x)} & \left(\frac{\beta(x)}{\alpha(x)}\right)^{1/2} \\ -\left(\frac{\beta(x)}{\alpha(x)}\right)^{1/2} & 0 \end{pmatrix}, \qquad F_0(x) := \begin{pmatrix} 0 \\ f(x) \end{pmatrix},$$

and the initial condition

$$U(0) = \begin{pmatrix} \alpha^{1/2}(0)\beta^{1/2}(0)u_0 \\ \alpha(0)u_1 \end{pmatrix}.$$

We now follow the reduction steps similar to those in Subsection 3.1. We write

$$A(x) = \left(\frac{\beta(x)}{\alpha(x)}\right)^{1/2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{[\alpha(x)\beta(x)]'}{2\alpha(x)\beta(x)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and note that we can diagonalise the first matrix with the help of the auxiliary constant matrix P introduced in (36). Denoting $V(x) := PU(x)$, we hence have, from (41),

$$V'(x) = B(x)V(x) + F_1(x),$$ (42)

with

$$B(x) := \begin{pmatrix} b_1(x) & 0 \\ 0 & \overline{b_1(x)} \end{pmatrix} + \frac{[\alpha(x)\beta(x)]'}{4\alpha(x)\beta(x)} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad F_1(x) := PF(x) = \begin{pmatrix} \frac{f(x)}{2^{1/2}} \\ i\frac{f(x)}{2^{1/2}} \end{pmatrix},$$

$$b_1(x) := i\left(\frac{\beta(x)}{\alpha(x)}\right)^{1/2} + \frac{[\alpha(x)\beta(x)]'}{4\alpha(x)\beta(x)},$$

and the initial condition

$$V(0) = PU(0) = \frac{1}{2^{1/2}} \begin{pmatrix} i\alpha^{1/2}(0)\beta^{1/2}(0)u_0 + \alpha(0)u_1 \\ \alpha^{1/2}(0)\beta^{1/2}(0)u_0 + i\alpha(0)u_1 \end{pmatrix}.$$

Introducing

$$W(x) := \begin{pmatrix} \exp\left(-\int_0^x b_1(\tau)\,d\tau\right) & 0 \\ 0 & \exp\left(-\int_0^x \overline{b_1(\tau)}d\tau\right) \end{pmatrix} V(x),$$

equation (42) transforms into

$$W'(x) = C(x)W(x) + G(x),$$ (43)

where

$$C(x) := \begin{pmatrix} 0 & c_1(x) \\ \overline{c_1(x)} & 0 \end{pmatrix}, \qquad c_1(x) := \frac{i[\alpha(x)\beta(x)]'}{4\alpha(x)\beta(x)} \exp\left(-2i\int_0^x \left(\frac{\beta(\tau)}{\alpha(\tau)}\right)^{1/2} d\tau\right),$$

$$G(x) := \begin{pmatrix} g_1(x) \\ i\overline{g_1(x)} \end{pmatrix}, \qquad g_1(x) := \frac{f(x)}{2^{1/2}} \exp\left(-\int_0^x \left[i\left(\frac{\beta(\tau)}{\alpha(\tau)}\right)^{1/2} + \frac{[\alpha(x)\beta(x)]'}{4\alpha(x)\beta(x)}\right] d\tau\right),$$

and the initial condition

$$W(0) = \begin{pmatrix} w_1^0 \\ i\overline{w_1^0} \end{pmatrix}, \qquad w_1^0 := \frac{1}{2^{1/2}}\left(i\alpha^{1/2}(0)\beta^{1/2}(0)u_0 + \alpha(0)u_1\right).$$

8

Here, in relating the first and the second components of the vector $G(x)$, and similarly $W(0)$, we employed the real-valuedness of $\alpha(x)$, $\beta(x)$, u_0, u_1 that was discussed in the beginning of this Subsection.

It remains to observe that system (43) is such that the matrix $C(x)$ and the vectors $G(x)$, $W(0)$ fit the assumptions discussed in Subsection 2.2.

3.3 Zakharov-Shabat system

It is well-known that solution of a spectral problem with the linear Schrödinger equation appears as an intermediate step in solving the Korteweg–de Vries (KdV) equation by using the inverse scattering transform. Zakharov-Shabat systems play the same role in the integrability of other nonlinear equations [1, p.10]. In particular, the Zakharov-Shabat system

$$\partial_x \left(\begin{array}{c} v_1(x,t) \\ v_2(x,t) \end{array} \right) = \left(\begin{array}{cc} -i\xi & q(x,t) \\ \overline{q(x,t)} & i\xi \end{array} \right) \left(\begin{array}{c} v_1(x,t) \\ v_2(x,t) \end{array} \right), \tag{44}$$

with $\xi \in \mathbb{R}$ being a spectral parameter, is a linear problem pertinent to the integration of the defocusing cubic nonlinear Schrödinger (NLS) equation

$$i\partial_t q(x,t) = \partial_x^2 q(x,t) - 2|q(x,t)|^2 q(x,t)$$

subject to the initial data $q(x,0) = q_0(x)$. We refer to [6] for more details on this matter.

We observe that by setting

$$W(x,t) := \left(\begin{array}{cc} e^{i\xi x} & 0 \\ 0 & e^{-i\xi x} \end{array} \right) \left(\begin{array}{c} v_1(x,t) \\ v_2(x,t) \end{array} \right),$$

Zakharov-Shabat system (44) immediately reduces to

$$\partial_x W(x,t) = \left(\begin{array}{cc} 0 & q(x)e^{2i\xi x} \\ \overline{q(x)}e^{-2i\xi x} & 0 \end{array} \right) W(x,t),$$

which is a system of the form (1).

3.4 Kubelka-Munk equations

Kubelka-Munk equations is a simple phenomenological model for computing reflection and transmission optical fluxes without solution of significantly more complicated radiative transfer equations [9]. Due to their simplicity, Kubelka-Munk equations have been popular in practice (in paper paint visibility, see e.g. [3]), they have been extensively studied from modelling viewpoint and several generalisations have been proposed [13, 14, 15, 16].

We consider the following model equations

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\begin{array}{c} F_+(x) \\ F_-(x) \end{array} \right) = \left(\begin{array}{cc} -K(x) - S(x) & S(x) \\ -S(x) & K(x) + S(x) \end{array} \right) \left(\begin{array}{c} F_+(x) \\ F_-(x) \end{array} \right), \tag{45}$$

where F_+, F_- are fluxes in positive and negative directions, and K and S are related to absorption and scattering, respectively. Note that, unlike in the classical model, we take here K, S to be dependent on the optical depth x rather than simply being constants. This generalisation is expected to be useful since constant scattering and absorption coefficients are known to be a considerable limitation of the Kubelka-Munk model [3, Sect. 4.5].

The procedure of reduction of (45) to (1) is similar to that performed in Subsection 3.1. Therefore, we shall omit any detailed calculation.

Let us write

$$A(x) := \begin{pmatrix} -K(x) - S(x) & S(x) \\ -S(x) & K(x) + S(x) \end{pmatrix} = S(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - [K(x) + S(x)] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and compute

$$B(x) := PA(x)P^{-1} = S(x) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - [K(x) + S(x)] \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

with P defined as in (36). Multiplying the both sides of (45) by P and introducing

$$V(x) := P \begin{pmatrix} F_+(x) \\ F_-(x) \end{pmatrix},$$

we obtain

$$V'(x) = B(x)V(x). \tag{46}$$

Furthermore, setting

$$W(x) := \begin{pmatrix} \exp\left(-i \int_0^x S(\tau)\,\mathrm{d}\tau\right) & 0 \\ 0 & \exp\left(i \int_0^x S(\tau)\,\mathrm{d}\tau\right) \end{pmatrix} V(x),$$

$$C(x) := \begin{pmatrix} 0 & c_2(x) \\ \overline{c_2(x)} & 0 \end{pmatrix}, \quad c_2(x) := -i[K(x) + S(x)] \exp\left(-2i \int_0^x S(\tau)\,\mathrm{d}\tau\right),$$

we arrive at

$$W'(x) = C(x)W(x), \tag{47}$$

which is a system of the form (1).

4   Conclusion

We have introduced a new scalar differential equation of the first order which is curious for two principal reasons. First, it is, in some sense, the simplest nonlinear ODE (either with or without a non-homogeneous term), with the nonlinearity being merely the complex conjugation. Second, this equation emerges, after appropriate reduction steps, in rather different physical contexts. Certainly, much more application areas can be identified (e.g. telegrapher's equations or Goldstein-Taylor model [5]), but already the context of the linear Schrödinger equation alone is a good enough motivation to further study the antilinear ODE $u'(x) = f(x)\overline{u(x)}$. For example, reduction of matrix-vector manipulations to those involving scalar quantities already provides a simplification in tedious constructions of asymptotic-numerical methods, cf. [2, 8]. Therefore, this new reformulation yields concrete practical advantages. We believe that theoretical aspects of the mentioned models could benefit from it, too. This might be achievable, for instance, through newly produced forms of the Prüfer transformation (which is typically used for studying Sturm-Liouville problems, see e.g. [11, Sect. 5.2]). Furthermore, it is important to identify classes of functions f for which the antilinear ODE can be solved in a closed form. This would, in particular, yield new solvable quantum mechanical potentials and sound-speed profiles important for generating reference solutions for verification of numerical methods for the wave propagation. Exploring this direction, we note that the Kubelka-Munk model context hints on the elementary exponential class of function $f(x)$ since system (45) with constant K and S is solvable explicitly. This can be generalised further since the form of the antilinear ODE is amenable to a treatment by integral transform methods (unlike other nonlinearities) typically compatible with an exponential function and combinations thereof. Finally, the form of the antilinear ODE calls for study of the possible connection with d-bar problems, see e.g. [7].

In this case, an appropriate extension of the equation to the complex plane may yield a formulation that eventually produces a closed-form solution due to numerous constructive results on Hilbert and Riemann-Hilbert problems.

Acknowledgements

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