The Limiting Amplitude Principle for the Wave Equation with Variable Coefficients

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The Limiting Amplitude Principle (LAP): the classical and the present setups

Time-domain problem

$$\begin{cases} \partial_t^2 u(x,t) + \mathcal{L}[u](x,t) = e^{-i\omega t}F(x), & x \in \mathbb{R}^d \backslash \overline{\Omega}_{int},\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega_{int},\ t > 0, \\ u(x,0) = 0, \quad \partial_t u(x,0) = 0, & x \in \mathbb{R}^d \backslash \overline{\Omega}_{int}. \end{cases} \tag{1}$$

LAP (if valid):

$$u(x,t) \underset{t\to\infty}{\longrightarrow} e^{-i\omega t} U(x)$$

(in some sense and with some rate)

Frequency-domain problem

$$\begin{cases} -\omega^2 U(x) + \mathcal{L}[U](x) = F(x), & x \in \mathbb{R}^d \backslash \overline{\Omega}_{int}, \\ U(x) = 0, & x \in \partial\Omega_{int}, \\ (\text{ Sommerfeld radiation condition }), & |x| \to \infty. \end{cases} \tag{2}$$

- Classical setting: $d = 3$, $\mathcal{L}[u] := -c_0^2 \Delta u$, $\Omega_{int} \subset \mathbb{R}^3$ open and bounded or $\Omega_{int} = \emptyset$, $F \in L^2(\mathbb{R}^3 \backslash \overline{\Omega}_{int})$ and compactly supported, $c_0 > 0$ constant.
- Present setting: $d \in \{1,2,3\}$, $\mathcal{L}[u] := -\beta^{-1}(x) \nabla \cdot (\alpha(x) \nabla u)$, $\Omega_{int} = \emptyset$, $F \in L^2(\mathbb{R}^d)$ and compactly supported,
  $\alpha, \beta \in C^\infty(\mathbb{R}^d)$, $\alpha(x), \beta(x) > 0$ non-trapping and $(\alpha(x) - \alpha_0)$, $(\beta(x) - \beta_0)$ are compactly supported for some constants $\alpha_0, \beta_0 > 0$.

Sommerfeld radiation condition: $\lim\limits_{|x|\to\infty} |x|^{\frac{d-1}{2}} \left[\partial_{|x|} U(x) - i\omega \sqrt{\beta_0/\alpha_0}\, U(x) \right] = 0.$

Our results

Main features:

- The LAP is proved for the wave equation that has non-constant coefficients and which is not necessarily in a divergent form. Besides the classical case $d = 3$, we also consider $d = 2$.
- The validity of the LAP is extended to the case $d = 1$ with an appropriate modification.
- The convergence in the LAP is quantified and is shown to be algebraic in time for $d \in \{2,3\}$ and exponential for $d = 1$.

Theorem (LAP: $d = 2$ and $d = 3$)

Let $d \in \{2,3\}$. Suppose that all assumptions of the present setting are satisfied. Let $u(x,t)$ and $U(x)$ be the solutions to (1) and (2), respectively. Then, for any bounded $\Omega \subset \mathbb{R}^d$, there exists a constant $C > 0$ such that

for $d = 2$, $t \geq 0$:

$$\left\| u(\cdot,t) - e^{-i\omega t} U \right\|_{H^1(\Omega)} + \left\| \partial_t u(\cdot,t) + i\omega e^{-i\omega t} U \right\|_{L^2(\Omega)} \leq C \frac{1 + \log(1+t^2)}{(1+t^2)^{\frac{1}{2}}},$$

for $d = 3$, $t \geq 0$:

$$\left\| u(\cdot,t) - e^{-i\omega t} U \right\|_{H^1(\Omega)} + \left\| \partial_t u(\cdot,t) + i\omega e^{-i\omega t} U \right\|_{L^2(\Omega)} \leq \frac{C}{(1+t^2)^{\frac{1}{2}}}.$$

Theorem (LAP: $d = 1$ - an extension of the principle)

Let $d = 1$. Suppose that all assumptions of the present setting are satisfied. Let $u(x,t)$ and $U(x)$ be the solutions to (1) and (2), respectively. Then, for any bounded interval $\Omega \subset \mathbb{R}$, there exist constants $u_\infty \in \mathbb{C}$, $\Lambda > 0$ and $C > 0$ such that, for $t \geq 0$,

$$\left\| u(\cdot,t) - e^{-i\omega t} U - u_\infty \right\|_{H^1(\Omega)} + \left\| \partial_t u(\cdot,t) + i\omega e^{-i\omega t} U \right\|_{L^2(\Omega)} \leq C e^{-\Lambda t}.$$

A practical application

One motivation to revisit the LAP and study rates of convergence is analysis of time-domain numerical methods.
In particular, such methods can be very attractive for high frequencies.
For example, by an appropriate treatment, computational cost can be effectively reduced by directing numerical efforts towards the wavefront area yet also resolving the wavefield in the whole domain of interest:

- Arnold, A., Geevers, S., Perugia, I., Ponomarev, D. An adaptive finite element method for high-frequency scattering problems with variable coefficients, arXiv:2103.02511, 2021.

Some ingredients of the proof

Most of the auxiliary results concern time decay of the solutions of the following initial-value problems:

$$\begin{cases} \partial_t^2 u(x,t) - \beta^{-1}(x) \nabla \cdot (\alpha(x) \nabla u(x,t)) = f(x,t), & x \in \mathbb{R}^d,\ t > 0, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), & x \in \mathbb{R}^d, \end{cases} \tag{3}$$

$$\begin{cases} \partial_t^2 v(x,t) - c_0^2 \Delta v(x,t) = 0, & x \in \mathbb{R}^d, \quad t > 0, \\ v(x,0) = v_0(x), \quad \partial_t v(x,0) = v_1(x), & x \in \mathbb{R}^d, \end{cases} \tag{4}$$

where $c_0 := \sqrt{\alpha_0/\beta_0}$.

Lemma (for the solution of (3))

Let $d \geq 2$, $f \equiv 0$, $u_0 \in H^3(\mathbb{R}^d)$, $u_1 \in H^2(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} \left(1 + |x|^2\right)^{d+1+\epsilon} \left(|u_0(x)|^2 + |u_1(x)|^2\right) dx < \infty$ with some $\epsilon > 0$.
Then, for any bounded $\Omega \subset \mathbb{R}^d$, there exists a constant $C > 0$ such that

$$\| u(\cdot,t) \|_{H^1(\Omega)} + \| \partial_t u(\cdot,t) \|_{L^2(\Omega)} \leq \frac{C}{(1+t^2)^{\frac{d-1}{2}}}, \quad t \geq 0.$$

Moreover, if $d = 1$, $f \equiv 0$, $u_0 \in H^1(\mathbb{R})$, $u_1 \in L^2(\mathbb{R})$ and are supported inside some $\Omega_0 \subset \mathbb{R}$. Then, for any bounded interval $\Omega \subset \mathbb{R}$, there exist a constant $C > 0$ such that

$$\| u(\cdot,t) - u_\infty \|_{H^1(\Omega)} + \| \partial_t u(\cdot,t) \|_{L^2(\Omega)} \leq C e^{-\Lambda t}, \quad t \geq 0,$$

where $u_\infty := \frac{1}{2} \int_{\Omega_0} \beta(x) u_1(x)\, dx$ and $\Lambda > 0$ can be estimated explicitly.

Lemma (for the solution of (3))

Let $d \in \{2,3\}$, $u_0 \equiv 0$, $u_1 \equiv 0$, $f \in C(\mathbb{R}_+, L^2(\mathbb{R}^d))$, $f(\cdot,t)$ is compactly supported for each $t \geq 0$, and, for some $C_0 > 0$,

$$\| f(\cdot,t) \|_{L^2(\mathbb{R}^d)} + \| \partial_t f(\cdot,t) \|_{L^2(\mathbb{R}^d)} \leq \frac{C_0}{(1+t^2)^{\frac{1}{2}}}, \quad t \geq 0.$$

Then, for any bounded $\Omega \subset \mathbb{R}^d$, there exists a constant $C > 0$ such that

for $d = 2$, $t \geq 0$: $\| u(\cdot,t) \|_{H^1(\Omega)} + \| \partial_t u(\cdot,t) \|_{H^1(\Omega)} \leq C \dfrac{1 + \log(1+t^2)}{(1+t^2)^{\frac{1}{2}}}.$

for $d = 3$, $t \geq 0$: $\| u(\cdot,t) \|_{H^1(\Omega)} + \| \partial_t u(\cdot,t) \|_{H^1(\Omega)} \leq \dfrac{C}{(1+t^2)^{\frac{1}{2}}}.$

Lemma (for the solution of (4))

Let $d \in \{2,3\}$ and v_0, v_1 are of some special form: either sufficiently localised and oscillatory or essentially outgoing. Then, for any bounded $\Omega \subset \mathbb{R}^d$, there exists a constant $C > 0$ such that

$$|v(x,t)| + \ldots + |\partial_t \Delta v(x,t)| \leq \frac{C}{(1+t^2)^{\frac{1}{2}}}, \quad x \in \Omega,\ t \geq 0.$$

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