On justification of a NLS model for laser beams in photopolymers

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Abstract

A nonstationary model that relies on the spatial nonlinear Schrödinger (NLS) equation with the time-dependent refractive index describes laser beams in photopolymers. We consider a toy problem, when the rate of change of refractive index is proportional to the squared amplitude of the electric field and the spatial domain is a plane. In the present work, we derive the NLS approximation from a two-dimensional quasi-linear wave equation and rigorously justify this model for appropriately small time intervals and smooth initial data. Numerical simulations illustrate the approximation result in the one-dimensional case.

1 Introduction

Mathematical models for laser beams in photochemical materials used in physical literature [2] are based on a spatial nonlinear Schrödinger (NLS) equation with a time-dependent refractive index. These models are normally derived from Maxwell equations using heuristic arguments and qualitative approximations (see e.g. [3]). Numerical simulations of such models are performed by experimentalists [1], [6] for theoretical explanations of complicated dynamics of laser beams in photopolymers. The complexity of the NLS equation modeling photochemical materials is related to the fact that the spatial coordinate in the direction of the beam propagation serves as an evolution time in the NLS equation, whereas the nonlinear refractive index depends slowly on the temporal coordinate. Physically, laser beams described by the NLS approximation induce waveguides in polymers, which affect the shape and dynamics of laser beams via nonlinear refractive index. In the present work we study how to justify a time-dependent NLS model derived from a toy model resembling the Maxwell equations. The toy model is written as a system of a two-dimensional quasilinear wave equation and an empirical relation for the change of the refractive index.

2 Model and results

A photopolymer occupies typically a half-space $z \geq 0$ and its face $z = 0$ is exposed to a laser beam. If the beam is localized in the x-direction and uniform in the y-direction, then the electric field has polarization in the y-direction with the amplitude E being y-independent, hence $\mathbf{E}(x,z,t) = (0, E(x,z,t), 0)$ is the electric field. The initial beam is assumed to be spatially wide-spreaded, small in amplitude, and monochromatic in time.

Neglecting polarization effects and uniform material losses, the electric field satisfies a one-dimensional quasilinear wave equation in the form

$$\partial_x^2 E + \partial_z^2 E - n^2 \partial_t^2 E = 0, \tag{1}$$

where n is referred to as the refractive index of the photopolymer. The refractive index n changes in time t because of the nonlinear effects induced by the squared amplitude of the electric field E.

Let us write the squared refractive index in the form $n^2 = 1 + m$ and assume that the rate of change of m is governed by the empirical relation

$$\frac{\partial m}{\partial t} = E^2. \tag{2}$$

The system (1)-(2) resembles approximation of a more complicated system of governing equations in physical literature [2]. We note that all physical constants in this system are normalized to unity.

Asymptotic solution to the system is given by the multi-scale expansion [5]

$$E(x,z,t) = \epsilon^{\frac{s+2}{2}} A(X,Z,T) e^{i\omega_0(z-t)} + \text{c.c.} \tag{3}$$

$$m(x,z,t) = \epsilon^2 M(X,Z,T), \tag{4}$$

where c.c. stands for complex conjugated terms, $X = \epsilon x$, $Z = \epsilon^2 z$, $T = \epsilon^s t$ are slow variables and $s \geq 2$.

If $s = 2$, the leading-order terms read as follows:

$$\partial_X^2 A + 2i\omega_0 (\partial_Z A + \partial_T A) + \omega_0^2 M A = 0 \tag{5}$$

and

$$\partial_T M = 2|A|^2, \tag{6}$$

which will be the subject of our studies.

If $s > 2$, at the leading order, we have the spatial NLS equation

$$\partial_X^2 A + 2i\omega_0 \partial_Z A + \omega_0^2 M A = 0. \tag{7}$$

Because M depends on T by means of the same equation (6), A depends on T implicitly in the spatial NLS equation (7). The system (6)-(7) was used in the previous works on photochemical materials (see review in [2]). While justification of the system (6)-(7) still remains an open problem, we focus on the system (5)-(6). We shall consider solutions of the original system (1)-(2) in an unbounded domain for $(x, z) \in \mathbb{R}^2$ supplemented by the initial conditions at $t = 0$. We hence work with the scaling $X = \epsilon x$, $Z = \epsilon^2 z$, $T = \epsilon^2 t$ and represent exact solution to the system (1)-(2) as

$$E(x, z, t) = \epsilon^2 \left(A(X, Z, T) e^{i\omega_0 (z-t)} + \text{c.c.} \right) + U(x, z, t) \tag{8}$$

and

$$m(x, z, t) = \epsilon^2 M(X, Z, T) + N(x, z, t), \tag{9}$$

where $U(x, z, t)$ and $N(x, z, t)$ are error terms to estimate. Feeding (8)-(9) into (1)-(2) and assuming validity of (5)-(6), we arrive at the system

$$\begin{aligned} \partial_x^2 U + \partial_z^2 U - \left(1 + \epsilon^2 M + N\right) \partial_t^2 U &= -\epsilon^2 R_1 N \\ &- \epsilon^6 R_2 \end{aligned} \tag{10}$$

and

$$\partial_t N = \epsilon^4 R_3 + \epsilon^2 R_4 U + U^2, \tag{11}$$

where $R_1, \ldots, R_4$ are some functions of A and its derivatives.

In our work [4], we establish local well-posedness of the systems (1)-(2) and (5)-(6), formulate a criterion for continuation of local solutions of (1)-(2) and obtain *a priori* energy estimates from residual equations derived from (10)-(11) by suitable near-identity transformations. As a main outcome, we have the following justification result for initial pulses lying in Sobolev Hilbert space $H^k\left(\mathbb{R}^2\right) := W^{k,2}\left(\mathbb{R}^2\right)$ with sufficiently high index k.

Theorem. *Given initial data $A_0 \in H^8\left(\mathbb{R}^2\right)$, let A, M be local solutions to the system (5)-(6) for $T \in [0, T_\infty)$, where $T_\infty > 0$ is the maximal existence time. There exist $\epsilon_0 > 0$ and $T_0 \in (0, T_\infty)$ such that for every $\epsilon \in (0, \epsilon_0)$ there is a unique solution E, m of the system (1)-(2) for $t \in \left[0, T_0/\epsilon^2\right]$ satisfying*

$$\sup_{t \in [0, T_0/\epsilon^2]} \left\| E - \epsilon^2 \left(A e^{i\omega_0 (z-t)} + c.c. \right) \right\|_{H^3(\mathbb{R}^2)} = \mathcal{O}\left(\epsilon^{5/2}\right),$$

$$\sup_{t \in [0, T_0/\epsilon^2]} \left\| m - \epsilon^2 M \right\|_{H^2(\mathbb{R}^2)} = \mathcal{O}\left(\epsilon^{5/2}\right).$$

3 Discussion and further challenges

We expect justificaton analysis of the NLS model (5)-(6) to be easily extended to the case of time evolution of a pulse in $\mathbb{R}^3$. But, in view of conventional experimental set-up, it would be more interesting to reformulate a problem in a half-space setting. However, justification happens to be problematic since the applied technique is incompatible with spaces in which we are able to prove well-posedness of the system (5)-(6). Moreover, in *a priori* energy estimates, there are non-vanishing boundary terms arising from integration by parts. In case of the spatial NLS model (6)-(7) for $X \in \mathbb{R}$, $Z \in \mathbb{R}_+$, the inapplicability of the energy method is obvious because $\|A\|_{L^2(\mathbb{R}\times\mathbb{R}_+)}$ becomes infinite due to conservation in Z of $L_X^2(\mathbb{R})$-norm of solution of the NLS equation (7).

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