Revisiting the limiting amplitude principle for the wave equation with variable coefficients

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Abstract

The limiting amplitude principle is a well-known result connecting the solution of the Helmholtz equation with the large-time behavior of time-dependent wave equations with a source term which is periodic in time. Motivated by numerical analysis of time-domain methods for stationary scattering problems in heterogeneous media, we quantify the solution convergence of such time-dependent wave equation towards the stationary solution, under some assumptions on the coefficients and the source term. We also generalise the formulation of the limiting amplitude principle to the one-dimensional setting where the classical statement of the principle is known to be violated.

Keywords: limiting amplitude principle, time-domain wave equation, Helmholtz equation, large time behavior

1 An introduction

It is a common engineering wisdom that the wave equation with a periodic-in-time source term yields a solution that stabilises for large times to the solution of the corresponding Helmholtz equation modulated by a time-harmonic factor. This link between time- and frequency-domain wave problems is known as the limiting amplitude principle. This principle has been been a subject of extensive research started nearly 70 years ago aiming at developing tools for the selection of a physically relevant unique solution of the Helmholtz equation in an unbounded domain. However, this viewpoint can be altered, and one can start with a problem governed by the Helmholtz equation (supplemented by classical Sommerfeld radiation conditions) which is known to admit a unique solution, and one can employ time-domain methods in order to efficiently find this solution. Despite a seemingly increased computational burden (due to the additional temporal dimension), some special time-domain methods such as [4,5] can be useful when wavenumber (frequency) is large and the original Helmholtz equation is difficult to solve numerically. In particular, in the time domain, one can take advantage of the presence of sharp wavefronts and resolve the problem on a suitably adapted mesh [1]. In the present communication, summarising [3], we focus on another aspect of evaluation of the efficiency of a time-domain approach to the Helmholtz equation. Namely, we are concerned with the speed of the convergence in time of the solution of the time-dependent wave equation to the solution of the underlying stationary problem.

2 Results

Given a frequency $\omega > 0$, a compactly supported source term $F \in L^2\left(\mathbb{R}^d\right)$ and non-trapping material parameters $\alpha_{\min} < \alpha(\mathbf{x}) \in C^\infty\left(\mathbb{R}^d\right)$, $\beta_{\min} < \beta(\mathbf{x}) \in C^\infty\left(\mathbb{R}^d\right)$ such that $\alpha(\mathbf{x}) = \alpha_0$, $\beta(\mathbf{x}) \equiv \beta_0$ for $\mathbf{x} \in \mathbb{R}^d \backslash \Omega_0$ with some bounded set $\Omega_0 \subset \mathbb{R}^d$ and constants $\alpha_{\min}$, $\beta_{\min} > 0$, we consider the Helmholtz equation, for $\mathbf{x} \in \mathbb{R}^d$,

$$-\nabla \cdot \left(\alpha(\mathbf{x}) \nabla U(\mathbf{x})\right) - \omega^2 \beta(\mathbf{x}) U(\mathbf{x}) = F(\mathbf{x}), \tag{1}$$

supplemented by the Sommerfeld condition

$$\lim_{|\mathbf{x}|\to\infty} \left[\partial_{|\mathbf{x}|} U(\mathbf{x}) - i\omega\sqrt{\beta_0/\alpha_0}\, U(\mathbf{x})\right] = 0.$$

The corresponding time-domain wave equation, for $\mathbf{x} \in \mathbb{R}^d$, $t > 0$,

$$\beta(\mathbf{x}) \partial_t^2 u(\mathbf{x}, t) - \nabla \cdot \left(\alpha(\mathbf{x}) \nabla u(\mathbf{x}, t)\right) = e^{-i\omega t} F(\mathbf{x}) \tag{2}$$

is supplemented by the initial conditions

$$u(\mathbf{x}, 0) = 0, \quad \partial_t u(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \mathbb{R}^d.$$

Our main result pertaining the convergence in time of the solution $u(\mathbf{x}, t)$ and its derivatives can be formulated as the following theo-

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rem which establishes and quantifies the limiting amplitude principle in the aforementioned setting.

Theorem 1 *Let U and u be the solutions to problems* (1) *and* (2), *respectively. Then, for any bounded domain $\Omega \subset \mathbb{R}^d$ and $t > 0$, the following estimates hold true.*

For $d = 3$:

$$\left\|u(\cdot,t) - e^{-i\omega t}U\right\|_{H^1(\Omega)} + \left\|\partial_t u(\cdot,t) + i\omega e^{-i\omega t}U\right\|_{L^2(\Omega)} \le \frac{C_1}{(1+t^2)^{1/2}}.$$

For $d = 2$:

$$\left\|u(\cdot,t) - e^{-i\omega t}U\right\|_{H^1(\Omega)} + \left\|\partial_t u(\cdot,t) + i\omega e^{-i\omega t}U\right\|_{L^2(\Omega)} \le \frac{C_2\left(1+\log\left(1+t^2\right)\right)}{(1+t^2)^{1/2}}.$$

For $d = 1$:

$$\left\|u(\cdot,t) - e^{-i\omega t}U - U_\infty\right\|_{H^1(\Omega)} + \left\|\partial_t u(\cdot,t) + i\omega e^{-i\omega t}U\right\|_{L^2(\Omega)} \le C_3 e^{-\Lambda t}.$$

Here, the constants C_1, C_2, $C_3 > 0$,

$$U_\infty := \frac{1}{2i\omega\sqrt{\alpha_0\beta_0}} \int_{\mathbb{R}} F(x)\,\beta(x)\,\mathrm{d}x,$$

and the decay rate $\Lambda > 0$ can be estimated explicitly.

Our proof of Theorem 1 is due to reduction towards several results concerning temporal decay for wave equations with sufficiently localised initial data or a source term. In particular, we build up on recent resolvent estimates from [6] and analysis of the one-dimensional wave equation [2].

Note that, in contrast to our setting, classical works (such as [7–10]) rarely quantify the limiting amplitude principle, and they deal with either constant-coefficient equations or variable-coefficient equations in the divergent form, with the main focus on the physical case $d = 3$. We also stress that our results show that, with a minor modification (by accounting for the additional constant term U_∞), the validity of the limiting amplitude principle extends to the case $d = 1$.

During the talk, I will briefly describe some of the ingredients of the proof, touch on sharpness and non-sharpness aspects of the estimates in Theorem 1 and discuss further possible extensions of our results.

References

[1] Arnold, A., Geevers, S., Perugia, I., Ponomarev, D., An adaptive finite element method for high-frequency scattering problems with smoothly varying coefficients, *Comp. Math. Appl.* **109**, pp. 1-14, 2022.

[2] Arnold, A., Geevers, S., Perugia, I., Ponomarev, D., On the exponential time-decay for the one-dimensional wave equation with variable coefficients, *arXiv:2201.04379*, pp. 1–16, 2022.

[3] Arnold, A., Geevers, S., Perugia, I., Ponomarev, D., On the limiting amplitude principle for the wave equation with variable coefficients, *arXiv:2202.10105*, 2022.

[4] Appelö, D., Garcia, F., Runborg, O., Waveholtz: Iterative solution of the Helmholtz equation via the wave equation, *SIAM J. Sci. Comp.* **42 (4)**, pp. A1950–A1983, 2020.

[5] Grote, M. J., Nataf, F., Tang, J. H., Tournier, P.-H. Parallel controllability methods for the Helmholtz equation, *Comp. Meth. Appl. Mech. & Eng.* **362**, 112846, 2020.

[6] Bouclet, J.-M., Burq, N., Sharp resolvent and time-decay estimates for dispersive equations on asymptotically Euclidean backgrounds, *Duke Math. J.* **170 (11)**, pp. 2575–2629, 2021.

[7] Eidus, D.M., The Principle of Limit Amplitude, *Russ. Math. Surv.* **24**, pp. 97–167, 1969.

[8] Ladyzhenskaya, O.A., On the principle of limit amplitude, *Uspekhi Mat. Nauk* **12**, pp. 161–164, 1957.

[9] Morawetz, C. S, The limiting amplitude principle, *Comm. Pure Appl. Math.* **3**, pp. 349–361, 1962.

[10] Odeh, F. M, Principles of Limiting Absorption and Limiting Amplitude in Scattering Theory - II - The Wave Equation in an Inhomogeneous Medium, *J. Math. Phys.* **6 (2)**, pp. 800–802, 1961.

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